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Equivariant Euler-Poincaré Characteristics and Tameness

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Introduction

In this paper we give a reasonably self-contained discussion of the Euler-Poincaré characteristics defined by Chinburg in the first part of [Ch1]. We show how these arise naturally when studying actions of a finite group G on coherent sheaves. More precisely, suppose $f : X \to Y$ is a tame G-covering of schemes which are proper and of finite type over a noetherian ring A. Let T be a coherent sheaf on X which has an action of G compatible with the action of G on O_X. (The construction we will give applies to bounded complexes of sheaves having coherent terms, but for simplicity we will assume in this introduction that T is a single sheaf.) One then has a naive coherent Euler-Poincaré characteristic

$$\chi(G,T) = \sum_i (-1)^i \cdot [H^i(X,T)]$$

in the Grothendieck group $G_0(AG)$ of all finitely generated AG-modules. We will show here how to lift $\chi(G,T)$ in a canonical way to a more refined Euler-Poincaré characteristic $\chi R\Gamma^+(f_*(T))$ in the Grothendieck group $CT(AG)$ of all finitely generated cohomologically trivial AG-modules. The natural forgetful homomorphism $CT(AG) \to G_0(AG)$ is in general neither surjective nor injective. Thus the existence of a canonical $\chi R\Gamma^+(f_*(T))$ in $CT(AG)$ mapping to $\chi(G,T)$ restricts the possibilities for $\chi(G,T)$ and also provides a more subtle invariant of T.

The motivation for defining $\chi R\Gamma^+(f_*(T))$ is to combine the insight into Euler-Poincaré characteristics arising from classical algebraic geometry and

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from the theory of the Galois module structure of rings of algebraic integers in tame extensions. For example, many results in algebraic geometry have to do with computing the multiplicities of irreducible representations of G in various cohomology groups. The connection of these multiplicities to character functions, such as Gauss sums, becomes clearer when one uses 'Hom-descriptions' of Grothendieck groups, as suggested by work on algebraic integers. Various results about rings of integers have close geometric counterparts, which in turn may suggest new approaches to studying rings of integers. As one example, Taylor's Theorem connects the stable isomorphism class of the ring of integers in a tame finite Galois extension of number fields to the root numbers appearing in the functional equations of L-series. In [Ch1, Ch2], some conjectural generalizations of Taylor's Theorem to tame G-coverings of schemes are discussed, and results in this direction are proved in the case of smooth projective varieties over a finite field. Over finite fields, one has an alternate approach using l-adic cohomology to proving the Galois Gauss sum congruences which are the deepest arithmetic part of the proof of Taylor's Theorem. (See [Ch1, Sect. 8].) This suggests looking for a new, geometric proof of Taylor's Theorem for rings of integers; at this time we know of no such proof.

In this paper we will focus on how to define $\chi R\Gamma^+(f_*(T))$. The generality of the definition makes it possible to consider examples of a widely varied nature. At the same time, we would like to stress that the definition provides a way of calculating $\chi R\Gamma^+(f_*(T))$.

We now give a quick survey of this paper. Sections 1 - 3 are mainly a summary of known results, definitions and examples which prepare the stage for the new results presented in Sections 4 and 5. In Section 1 we recall the two examples which have motivated essentially all research on Galois module theory. Section 2 is an exposition of some well-known applications of Euler-Poincaré characteristics. In Section 3 we recall the notions of G-coverings and Galois G-coverings of schemes and of quasicoherent G-sheaves. In Section 4 we introduce what we call numerically tame G-coverings. These coverings are more general than the ones considered in [Ch1] and [Ch2], and it is for them that we may define $\chi R\Gamma^+(f_*(T))$. The construction of $\chi R\Gamma^+(f_*(T))$ is carried out in Section 5. The Appendix contains a proof –based on Abhyankar's Lemma– that G-coverings which are tame in codimension 1 are numerically tame.

1. Two basic examples

The following examples are included to give the reader an idea of the kind of information on the Galois module structure of G-coverings one should expect to extract from a description of the classes defined in Section 5.

1.1. Galois structure of differentials and generalizations. One of the first uses of character theory outside group theory was Hecke's determination of the Galois module structure of the complex vector space V of cusp forms of weight 2 and level a prime number p. Hecke's goal was to simplify the problem of studying such cusp forms by decomposing V into isotopic components under the natural action of $PSL(2,p)$ (see [He, 1-2]). As is well known V can be identified with the space $H^0(X,\Omega^1)$ of holomorphic differentials on the Riemann surface X which is the compactification of the orbit space of the action by the congruence subgroup $\Gamma(p)$ on the upper half plane (see [L], [Sh, 2.17]): so Hecke was considering the covering $X \to Y = \mathbf{P}^1$ with the group $PSL(2,p)$ acting on X and Ω^1. In [C-W] [W2] Chevalley and Weil generalize part of Hecke's work and deal with G-coverings $X \to Y$ of compact Riemann surfaces with an arbitrary finite group G as group of automorphisms. They give a formula for the multiplicity m_χ of any irreducible character χ of G in $H^0 = H^0(X,\Omega^1)$ in terms of the genus and ramification data. We observe that this can interpreted as follows. The space H^0 is a finite dimensional module over the semisimple algebra $\mathbf{C}G$ and hence determines a class $[H^0]$ in the Grothendieck group $K_0(\mathbf{C}G) = R(G)$ of (projective) $\mathbf{C}G$-modules – this is of course nothing but the group of virtual characters of G. Since $R(G)$ is the free abelian group on the irreducible characters of G, there is a natural isomorphism from $R(G)$ to $\mathrm{Hom}(R(G),\mathbf{Z})$; this isomorphism sends $[H^0]$ to the homomorphism which on an irreducible character χ of G takes the value m_χ.

These results have been considerably generalized in two directions. The first generalization concerns equivariant Euler-Poincaré characteristics of coherent sheaves other than the sheaf of differentials, mainly for covers of varieties which are smooth and proper over an algebraically closed field (see for example [G-G-H], [E-L], [N,1-3], [V-M]). The second generalization concerns Euler-Poincaré characteristics of sheaves for the étale topology. The formulas that have been obtained in this case are generalizations of Weil's interpretation of the fact that in characteristic 0 the determination of the G-action on $H^0(X,\Omega^1)$ determines the action on the space of harmonic forms: this space is dual to the first homology space of X and so we should study the Tate module of the Jacobian of the curve, which plays the role of the first homology group for the étale topology. This leads to a formula for the Artin conductor attached to a G-covering in positive characteristic (see [W1, p. 79],[S1] [Mi],[R]).

1.2. Galois structure of rings of integers. Let N/K be a finite Galois extension of number fields with group G. The ring of integers O_N is a $\mathbf{Z}G$-module and one can show that it is projective if and only if the extension is tamely ramified. Suppose N/K is tamely ramified, so there is a class

(O_N) in the *reduced* Grothendieck group $Cl(\mathbf{Z}G)$ of projective $\mathbf{Z}G$-modules (of rank 0). By results of Fröhlich one can describe $Cl(\mathbf{Z}G)$ in terms of homomorphisms from the group of virtual characters $R(G)$ into the group of idèles of an algebraic closure $(\mathbf{Q}^c)$ of $\mathbf{Q}$. The class (O_N) is then shown to be determined by a homomorphism defined *via* the Galois-Gauss sums of the characters of G (see [F, Ch. 1]).

We observe that most of the results in (1.1) deal with the determination of the actual isomorphism class of the modules involved whereas in (1.2) the emphasis is shifted to the determination of the *stable* isomorphism class. In both cases, however, what one really does when computing the classes is to determine a homomorphism on the virtual characters.

2. Applications of Euler-Poincaré characteristics

In this section we recall some elementary applications of Euler-Poincaré characteristics and of their equivariant generalizations.

2.1. The Riemann Problem. Let X be a projective variety over an algebraically closed field k. The Riemann problem has do to with determining the dimension of the k-vector space $H^0(X,T)$ of global sections of a coherent sheaf T on X. The classical approach to this problem has two steps: (a) Find an expression for the *Euler Poincaré characteristic*

$$\chi(T) = \sum_{i=0}^{\infty}(-1)^i \cdot dim_k H^i(X,T)$$

by means of Generalized Riemann-Roch Theorems, and (b) Prove a Vanishing Theorem which asserts that under suitable hypotheses, $H^i(X,T) = 0$ for $i > 0$. For such T, (a) gives an expression for $\chi(T) = dim_k H^0(X,T)$.

If one has compatible actions of a group G on X and on T, then one can refine the Riemann problem by asking for the kG-module structure of $H^0(X,T)$, as in Sect. (1.1). In step (a), it is then natural to consider the Euler-Poincaré characteristic

$$\chi(G,T) = \sum_{i=0}^{\infty}(-1)^i \cdot [H^i(X,T)]$$

in $G_0(kG)$. (Recall that $G_0(k) = K_0(k)$ can be identified to the ring of integers via the dimension map.) If one cannot accomplish step (b), one may be able nonetheless to restrict the possibilities for the class of $H^i(X,T)$ in $G_0(kG)$ for $i > 0$. For example, one might be able to show that various irreducible representations cannot occur in $H^i(X,T)$ for $i > 0$. In this way one may still deduce from $\chi(G,T)$ information about the class of $H^0(X,T)$ in $G_0(kG)$.