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METRIC PROPERTIES OF ALGORITHMS INDUCING LÜROTH SERIES EXPANSIONS OF LAURENT SERIES

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1. Introduction

Recently the present authors [**8**] introduced and studied some properties of various unique expansions of formal Laurent series over a field F, as the sums of reciprocals of polynomials, involving "digits" $a_1, a_2, \ldots$ lying in a polynomial ring $F[X]$ over F. In particular, one of these expansions (described below) turned out to be analogous to the so-called *Lüroth* expansion of a real number, discussed in Perron [**15**] Chapter 4.

In a partly parallel way, Artin [**1**] and Magnus [**11,12**] had earlier studied a Laurent series analogue of simple *continued fractions* of real numbers, involving "digits" $x_1, x_2, \ldots$ in a polynomial ring as above. In addition to sketching elementary properties of an n-dimensional "Jacobi–Perron" variant of this, Paysant–Leroux and Dubois [**13, 14**] also briefly outlined certain "metric" theorems analogous to some of Khintchine [**7**] for real continued fractions, in the case when F is a finite field. The main aim of this paper is to state or derive some similar metric or ergodic results for the Laurent series Lüroth-type expansion referred to above. (These results were introduced at the Geneva conference by the first-named author, and are *partly* based on his forthcoming paper [**9**]. For analogous results concerning Lüroth expansions of *real* numbers, see Jager and de Vroedt [**5**] and Salát [**16**].)

In order to explain the conclusions, we first fix some notation and describe the inverse–polynomial Lüroth–type representation to be considered:

Let $\mathcal{L} = F((z))$ denote the field of all *formal Laurent series* $A = \sum_{n=v}^{\infty} c_n z^n$ in an indeterminate z, with coefficients c_n all lying in a given field F. Although the main case of importance usually occurs when F is the field $\mathbb{C}$ of complex numbers, certain interest also attaches to other ground fields F and most of the results of [8] hold for arbitrary F. It will be convenient to write $X = z^{-1}$ and also consider the ring $F[X]$ of *polynomials* in X, and the field $F(X)$ of *rational functions* in X, with coefficients in F.

If $c_v \neq 0$, we call $v = v(A)$ the *order* of A above, and define the *norm* (or *valuation*) of A to be $\|A\| = q^{-v(A)}$, where initially $q > 1$ may be an arbitrary constant, but *later will be chosen* as $q = \operatorname{card}(F)$, if F is finite. Letting $v(0) = +\infty$, $\|0\| = 0$, one then has (cf. Jones and Thron [6] Chapter 5):

$$\|A\| \geq 0 \text{ with } \|A\| = 0 \text{ iff } A = 0,$$

(1.1)

$$\|AB\| = \|A\| \cdot \|B\|, \text{ and}$$

$\|\alpha A + \beta B\| \leq \max(\|A\|, \|B\|)$ for non–zero $\alpha, \beta \in F$, with equality when $\|A\| \neq \|B\|$. By (1.1), the norm $\|\ \|$ is non–Archimedean, and it is well known that $\mathcal{L}$ forms a *complete* metric space relative to the metric ρ such that $\rho(A, B) = \|A - B\|$.

In terms of the notation $X = z^{-1}$ above, we shall make frequent use of the polynomial $[A] = \sum_{v \leq n \leq 0} c_n X^{-n} \in F[X]$, and refer to $[A]$ as the *integral part* of $A \in \mathcal{L}$. Then $v = v(A)$ is the *degree* $\deg[A]$ of $[A]$ relative to X, and the same function $[\ ]$ was used by Artin [1] and Magnus [11, 12] for their continued fractions. (For a recent application of Artin's algorithm, F finite, see Hayes [4].)

Given $A \in \mathcal{L}$ now note that $[A] = a_0 \in F[X]$ iff $v(A_1) \geq 1$ where $A_1 = A - a_0$. As in [8], if $A_n \neq 0 (n > 0)$ is already defined, we then let $a_n = \left[\frac{1}{A_n}\right]$ and put $A_{n+1} = (a_n - 1)(a_n A_n - 1)$. If some $A_m = 0$ or $a_n = 0$, this recursive process stops. It was shown in [8] that this algorithm leads to a finite or convergent (relative to ρ) *Lüroth–type* series expansion

$$A = a_0 + \frac{1}{a_1} + \sum_{r \geq 2} \frac{1}{a_1(a_1 - 1) \ldots a_{r-1}(a_{r-1} - 1) a_r}, \tag{1.2}$$

where $a_r \in F[X]$, $a_0 = [A]$, and $\deg(a_r) \geq 1$ for $r \geq 1$. Furthermore this expansion is *unique* for A subject to the preceding conditions on the "digits" a_r.

If I denotes the ideal in the power series ring $F[[z]]$, consisting of all power series x such that $x(0) = 0$, then another way of looking at this expansion algorithm is in terms of operators $a : I - \{0\} \to F[X]$, $T : I \to I$ such that $a(x) = \left[\frac{1}{x}\right]$, $T(0) = 0$ and otherwise $T(x) = \big(a(x) - 1\big)\big(xa(x) - 1\big)$. Then, for $x = A_1 \in I$, $a_1 = a_1(x) = a(x)$, and more generally $a_n = a_n(x) = a_1(T^{n-1}x)$ if $0 \neq T^{n-1}x \in I$.

From now on it will be *assumed* that $F = \mathbb{F}_q$ is a *finite* field with exactly q elements. For that case it was shown in [**9**] that $T : I \to I$ is *ergodic* relative to the Haar measure μ on I such that $\mu(I) = 1$, and that this fact implies in particular:

Theorem 1. (i) *For any given polynomial $k \in \mathbb{F}_q[X]$, $deg(k) \geq 1$, and all $x \in I$ outside a set of Haar measure 0, the digit value k has asymptotic frequency*

$$\lim_{n\to\infty} \frac{1}{n} \#\{r \leq n : a_r(x) = k\} = \|k\|^{-2} = q^{-2\deg(k)}.$$

(ii) *For all $x \in I$ outside a set of Haar measure 0 there exists a single asymptotic mean–value*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \deg\big(a_r(x)\big) = \frac{q}{q-1}.$$

(iii) *For all $x \in I$ outside a set of Haar measure 0,*

$$\|x - w_n\| = q^{\left(-\frac{2q}{q-1} + o(1)\right)n} \quad \text{as } n \to \infty,$$

where

$$w_n = w_n(x) := \sum_{r=1}^{n} \frac{\lambda_{r-1}}{a_r}, \; \lambda_0 = 1, \; \lambda_r = \frac{1}{a_1(a_1 - 1)\ldots a_r(a_r - 1)}.$$

Regarding (iii), a similar but more elementary algebraic conclusion [**8**] states that

$$\|x - w_n\| \leq q^{-2n-1} \quad \text{for all } x.$$

Our main aim in the present article will be to state and prove various further metric results concerning polynomial "digits" $a_r(x)$ and their limiting distributions.

2. Sharper Metric Conclusions

A useful description of the Haar measure μ on I is given in Sprindžuk [**17**]. In particular $\mu(C) = q^{-r}$ for any "circle", "disc" or "ball"

$$C = C(x, q^{-r-1}) := \left\{ y \in \mathcal{L} : \|x - y\| \leq q^{-r-1} \right\}.$$

Using this, the proof of Theorem 1 (i) in [**9**] includes:

$$\mu\left\{x \in I : a_r(x) = k\right\} = \|k\|^{-2} \tag{2.1}$$

for any $k \in \mathbb{F}_q[X]$, $\deg(k) \geq 1$, and $r \geq 1$, and

(2.2) the Lüroth–type digits $a_r(x)$ define identically–distributed independent random variables a_r relative to μ on I.

More precisely, by Theorem 3.16 and the law of the iterated logarithm (Theorem 3.17) in Galambos [**3**], we obtain:

Theorem 2. *Let* $A_{n,k}(x) = \#\{r \leq n : a_r(x) = k\}$. *Then for almost all* $x \in I$

$$\limsup_{n \to \infty} \frac{A_{n,k}(x) - n\|k\|^{-2}}{\sqrt{n \log \log n}} = \sqrt{2\|k\|^{-2}(1 - \|k\|^{-2})}.$$

Further, for any real s,

$$\lim_{n \to \infty} \mu \left\{ x \in I : A_{n,k}(x) - n\|k\|^{-2} < \frac{s}{\|k\|} \sqrt{\frac{n}{(1 - \|k\|^{-2})}} \right\}$$
$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{s} e^{-u^2/2} du.$$

Next we note that Theorem 1 (ii) is equivalent to the existence of a *Khintchine–type* constant

$$\lim_{n \to \infty} \left\| a_1(x) a_2(x) \ldots a_n(x) \right\|^{1/n} = q^{q/(q-1)} \ a.e.$$

This conclusion can be refined to: