Astérisque

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Astérisque, tome 209 (1992), p. 311-320
<http://www.numdam.org/item?id=AST_1992__209__311_0>

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K-UNIRATIONALITY OF CONIC BUNDLES OVER LARGE ARITHMETIC FIELDS

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In the study of rational surfaces a major role is played by Del Pezzo surfaces and by conic bundles over curves. There are some interesting open questions about their properties. Most important among them is the problem of K-unirationality for conic bundles having a K-rational point. For a more precise exposition of this problem we need some definitions and conventions.

Let K be a field of characteristic $\neq 2$, X an absolutely irreducible variety defined over K, and $\overline{K}$ the algebraic closure of K.

Recall that X is said to be K-rational (respectively, K-unirational) if its function field $K(X)$ is (respectively, is contained in) a purely transcendental extension of K. One says that X is rational if $\overline{X} = X \times_K \overline{K}$ is $\overline{K}$-rational.

DEFINITION 1. A rational K-surface X is called a *conic bundle* over a rational curve C if there exists a K-morphism $f\colon X \to C$ whose generic fibre is a rational curve.

The problem of K-unirationality for conic bundles without a K-rational point clearly has a negative solution. So the following question is natural: are rational conic bundle surfaces with a K-rational point K-unirational? This question is not only of arithmetic interest. Its algebraic significance lies in the connection with the problem of existence of splitting fields of special type for some quaternion algebras. More precisely, V. A. Iskovskikh [8] has established that a conic bundle is K-unirational if and only if the corresponding quaternion algebra over a K-rational field has a K-rational splitting field (here a 'K-rational field' means some rational function field $K(z)$). Thus our K-unirationality question can be formulated as a problem in the theory of central simple algebras.

Further, this problem generalizes to arbitrary finite-dimensional central simple algebras. For a strict formulation we shall need the following definitions.

Let v be a valuation (or a place) of a field F. We shall denote by F_v the completion of F with respect to v (or at v).

DEFINITION 2. Let A be a finite-dimensional central simple algebra over a K-rational field L. Then one says that A *has a K-rational point* if there exist two elements, $k \in K$ and $x \in L$, such that $L = K(x)$ and the algebra $A \otimes_{K(x)} K(x)_{(x-k)}$ (where $(x-k)$ denotes the valuation of $K(x)$ corresponding to $x - k$ with trivial restriction to K) is trivial (i.e. a total matrix algebra over $K(x)_{(x-k)}$).

With the above notation (and definitions), the problem of existence of rational splitting fields for quaternion algebras generalizes as follows.

PROBLEM. *Let A be a finite-dimensional central simple algebra over a K-rational field, and suppose it has a K-rational point. Does A have a K-rational splitting field ?*

For some classes of fields this problem has a positive answer. This is trivial in the case of an algebraically closed field K. Actually, in this case, $K(x)$ is a C_1-field (for definitions see e.g. [15]), and the Brauer group of $K(x)$ is trivial (see [15]). Hence any extension of $K(x)$ (in particular, any K-rational field) is a splitting field for any central simple algebra over $K(x)$. The first nontrivial case is that of local fields (i.e. real closed and p-adically closed fields). The case where $K = \mathbb{R}$ was first considered by Iskovskikh [8]. Real closed fields were considered later by the author. As to p-adically closed fields, the author [18] proved that the above problem has a positive solution for Henselian fields K and hence for p-adically closed fields (see [14]). The next natural case for consideration is when K is 'pseudo-closed'. The aim of this paper is an exposition of results on the above problem in this case and of the analogous result for the so-called large arithmetic fields. These results were obtained recently by Yu. Drakokhrust and the author.

The author would like to thank the referee for some useful suggestions (see the Appendix).

§1. The case of pseudo-closed fields

In this section we shall prove that if a central simple algebra over a K-rational field has a K-rational point, then it has a K-rational splitting field, provided K is 'pseudo-closed'. We recall some definitions.

DEFINITION 3. A field K is said to be *formally real* if it has at least one ordering (for details see e.g. [11] and [12]).

DEFINITION 4. A field K is said to be *real closed* if it is formally real and does not admit any proper formally real algebraic extension.

Any real closed field has a unique ordering, and any formally real field is contained in some real closed one. Moreover, if L_1 and L_2 are two real closed algebraic field extensions of K whose orderings induce the same ordering v on K, then L_1 and L_2 are K-isomorphic and one says that L_1 is a real closure of K (with respect to v). Let us denote by S_K the set of all orderings on K (we do not rule out the possibility that S_K may be empty) and by K_v the real closure of K for each $v \in S_K$.

DEFINITION 5. A field K is called *pseudo-real closed (prc)* if any absolutely irreducible affine K-variety X has a K-rational point if and only if it has a simple K_v-point for every $v \in S_K$.

These definitions imply that the class of pseudo-algebraically closed (pac-) fields K is contained in that of prc-fields. This is a case where S_K is empty. (Pac-fields were introduced by J. Ax and have been systematically investigated in [4], [9], and [17]. As to prc-fields, see e.g. [13].)

The class of p-adically closed fields can be defined in a similar fashion.

DEFINITION 6. Let K be a field of characteristic zero with valuation v, valuation ring Ω, and maximal ideal $\Sigma \subset \Omega$. Suppose the field Ω/Σ is of characteristic p and $\dim_{\mathbb{Z}/p\mathbb{Z}} \Omega/\Sigma = d$. Then K is called a *p-valued field* of p-rank d.

DEFINITION 7. Let K be a p-valued field of p-rank d. Then K is said to be *p-adically closed* if K does not admit any proper p-valued algebraic extension with the same p-rank.

If K is a p-valued field with valuation v, then there exists a maximal p-valued algebraic extension of K having the same p-rank. Any such field K_v is called a *p-adic closure* of K.

DEFINITION 8. Let K be a p-valued field and let M_K be the set of all non-K-isomorphic p-adic closures of K. Then K is said to be *pseudo-p-adically closed (ppc)* if every absolutely irreducible affine K-variety has a K-rational point provided it has a simple L-rational point for every element L of M_K.

REMARK 1. The Brauer groups of prc-fields can be finite but, contrary to the case of pac- and real closed fields, their orders are not uniformly bounded and can even be infinite.

From now on, pac-, prc- and ppc-fields will be called *pseudo-closed fields* for short. The main result of this section is as follows.

THEOREM 1. *Let K be a pseudo-closed field. If a finite-dimensional central simple algebra A over a K-rational field $K(x)$ has a K-rational point, then it has a K-rational splitting field.*

REMARK 2. In the case of pac-fields K, this theorem was proved earlier by I. I. Voronovich [16].

Before proving the theorem, it is convenient to formulate the main result of [18].

THEOREM (*). *Let A be a central simple (finite-dimensional) algebra over a K-rational field $K(x)$. We assume that A has a K-rational point. If K is Henselian and its characteristic does not divide the index of A, then A has a K-rational splitting field $K(z)$. Furthermore,*

$$x = h(z)/\pi,$$

where $h(z) \in K[z]$ is an irreducible monic polynomial and π is a suitable element in the valuation ideal of K. In addition, a root α of $h(z)$ generates a Galois extension of K, and $K(\alpha)(x)$ is a splitting field of A.

Proof of Theorem 1. Let A be a central division algebra over $K(x)$ and let $\dim_{K(x)} A = n^2$. We denote by P the algebraic closure of $K(t)$ in $K\langle t\rangle$, where $K\langle t\rangle$ is the field of formal power series in t over K. Then P is a Henselian field [2]. By Theorem (*) the algebra $B = A \otimes_{K(x)} P(x)$ has a P-rational splitting field $P(z)$ such that

$$x = h(z)/t^m.$$

Here n divides m, and $h(z) \in K[z]$ is an irreducible monic polynomial with the property: if $h(\alpha) = 0$ then $K(\alpha)$ is a Galois extension of K and $K(\alpha)(x)$ is a splitting field of A.