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ON RIEMANN ZETA-FUNCTION AND ALLIED QUESTIONS

K. RAMACHANDRA

TO PROFESSOR R. BALASUBRAMANIAN
ON HIS FORTIETH BIRTHDAY

§ 1. **Introduction.** I will divide my lecture as follows.

§ 1. Introduction

§ 2. Conjectures 1 and 2.

§ 3. Conjecture 3 and a further question.

§ 4. Balasubramanian-Ramachandra (sufficient) condition for the validity of Conjecture 3.

§ 5. The Balasubramanian-Ramachandra condition is not always satisfied (Counterexample : a power of the Kahane series).

§ 6. Progress towards Conjectures 1 and 2 (Titchmarsh Series-I, Weak Titchmarsh Series, and Titchmarsh Series-II)

§ 7. Applications of Theorems 1 to 4 of § 6.

§ 8. Consequences of Conjectures 1 and 2 without Riemann Hypothesis.

§ 9. Further Conjectures (which would follow from Conjectures 1 and 2 and Riemann Hypothesis).

As will be seen in § 7 Conjectures 1 and 2 are more intimately connected with the Riemann zeta-function (and also with zeta and L-functions of algebraic number fields, zeta-functions of ray class fields and so on). But we emphasise only on the Riemann zeta-function.

§ 2. **Conjectures 1 and 2.** For all $N \geq H \geq 1000$ and all N-tuples $a_1 = 1, a_2, \cdots, a_N$ of complex numbers prove (or disprove!) that

Conjecture 1.

$$\frac{1}{H}\int_0^H \Big|\sum_{n\leq N} a_n n^{it}\Big|\, dt \geq 10^{-1000}, \tag{1}$$

Conjecture 2.

$$\frac{1}{H}\int_0^H \Big|\sum_{n\leq N} a_n n^{it}\Big|^2 dt \geq (\log H)^{-1000} \sum_{n\leq M} |a_n|^2 \tag{2}$$

where $M = H(\log H)^{-2}$.

Remark 1. We can formulate conjectures where RHS in (1) and (2) are bigger than the present ones. For example we can replace RHS of (2) by

$$C_1 \sum_{n\leq C_2 H} |a_n|^2$$

where $C_1 > 0$ and $C_2 > 0$ are numerical constants (in fact with any $C_1 < 1$).

Remark 2. Let $1 = \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ be any sequence of real numbers with $\lambda_{n+1} - \lambda_n$ bounded both above and below. Then we can formulate more general conjectures where $\sum_{n\leq N} a_n n^{it}$ is replaced by $\sum_{n\leq N} a_n \lambda_n^{it}$.

§ 3. **Conjecture 3 and a further question.** Let $a_1 = \lambda_1 = 1$, $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$, $\frac{1}{C} \leq \lambda_{n+1} - \lambda_n \leq C$, $\{a_n\}$ $(n = 1, 2, 3, \cdots)$ a sequence of complex numbers such that $F(s) = \sum_{n=1}^{\infty} a_n \lambda_n^{-s}$ is convergent for some $s = \sigma + it$ in the complex plane and can be continued analytically in $(\sigma \geq \beta, T \leq t \leq 2T)$ where β is a positive constant $< \frac{1}{2}$ and there the maximum of $|F(s)| \leq T^A$ where A and C are positive constants ≥ 1. Suppose $\sum_{n\leq x} |a_n|^2 \gg_{\varepsilon} x^{1-\varepsilon}$ for every $\varepsilon > 0$ and every $x \geq 1$. Then prove (or disprove) the

Conjecture 3. *The number of zeros of* $F(s)$ *in* $(\sigma \geq \beta, T \leq t \leq 2T)$ *is*

$$\gg T \log T. \tag{3}$$

Remark. A simple application of Jensen's Theorem (see $[T]_1$, page 125) shows that the number of zeros of $F(s)$ in $(\sigma \geq (\beta + \frac{1}{2})/2,\ T + D \leq t \leq 2T - D)$ is $\ll T \log T$, provided D is a large positive constant. Thus speaking roughly, the order of magnitude of the number of zeros in question is $T\log T$ (if the Conjecture 3 is true).

A further question. *Let $\beta(T) < \frac{1}{2}$ and $\beta(T) \to \frac{1}{2}$ as $T \to \infty$. Then study the lower bound for the number of zeros of $F(s)$ in*

$$(\sigma \geq \beta(T), T \leq t \leq 2T). \tag{4}$$

§ 4. **Balasubramanian-Ramachandra (sufficient) condition for the validity of Conjecture 3.**

Sufficient Condition. *There should exist a positive constant $\delta < \frac{1}{2} - \beta$ such that*

$$\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^2\, dt \ll \left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|\, dt\right)^2. \tag{5}$$

Remark 1. We will show in this section that the condition (5) is sufficient for the validity of Conjecture 3. (Note that the inequality which is the opposite of (5) is always true). From our proof it will be plain that the condition

$$\left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^g\, dt\right)^h \ll \left(\frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|^h\, dt\right)^g \tag{5'}$$

for some constants g, h with $0 < h < g$ is also sufficient.

Remark 2. A class of examples of $F(s)$ satisfying the condition (5) were studied in $[\mathrm{BR}]_3, [\mathrm{BR}]_4, [\mathrm{R}_5]$, and to some extent $[\mathrm{BR}]_5$. We content here by saying that a simple example of $F(s)$ satisfying (5) is $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ where $\chi(n)$ is any sequence of complex numbers with $\sum_{n\leq x}\chi(n) = O(1)$. For the series $\zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ where $\sum_{n\leq x}\chi(n) = O(x^{\frac{1}{2}-\eta})$ for some constant $\eta > 0$ and further $\chi(n) = O(1)$ the lower bound for the number of zeros in $(\sigma \geq (1-\eta)/2, T \leq t \leq 2T)$ is $\gg T(\log T)(\log\log T)^{-1}$. Both these results have been generalised to generalised Dirichlet series of the form $F(s) = \sum_{n=1}^{\infty}(a_n b_n \lambda_n^{-s})$ where $\{a_n\}$ and $\{b_n\}$ are somewhat general sequences of complex numbers and $\{\lambda_n\}$ is a somewhat general sequence of real numbers (see $[\mathrm{BR}]_3, [\mathrm{BR}]_4, [\mathrm{R}]_5, [\mathrm{BR}]_5,$). Less satisfactory results than Conjecture 3 were obtained in $[\mathrm{R}]_3$, and $[\mathrm{R}]_4$.

Remark 3. For suitable $\beta(t)$ the lower bounds for the number of zeros in (4) of $F(s)$ were studied in $[\mathrm{R}]_6, [\mathrm{BR}]_6$, and $[\mathrm{BR}]_7$. For example in $[\mathrm{R}]_6$, it was shown in great generality that $F(s)$ has $\gg T^{1-\varepsilon}$ zeros in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)^{-1}, T \leq t \leq 2T)$. In $[\mathrm{BR}]_6$, the functions mentioned in Remark 2 were considered. For example it was proved that the number of zeros of

$F(s) = \zeta(s) + \sum_{n=1}^{\infty}(\chi(n)n^{-s})$ $\left(\text{with } \chi(n) = O(1) \text{ and } \sum_{n\leq x}\chi(n) = O\left(x^{\frac{1}{2}-\delta}\right)\right)$ in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)^{\frac{3}{2}}(\log T)^{-\frac{1}{2}}, T \leq t \leq 2T)$ is $\gg T(\log\log T)^{-1}$. In $[\mathrm{BR}]_7$, it was shown that (for $(\log T)^C \leq H \leq T$ where C is a large constant) in $(\sigma \geq \frac{1}{2} - C_0(\log\log T)(\log H)^{-1}, T \leq t \leq T+H)$ $\zeta(s)$ has $\gg H(\log\log\log T)^{-1}$ zeros. The interesting fact was that only the Euler product and the analytic continuation (in fact for the lower bound $\gg \cdots$ the upper bound $\log|F(s)| \ll (\log T)^C$ is enough)

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=1}^{\infty}\left(n^{-s} - \int_n^{n+1} u^{-s}du\right), (\sigma > 0),$$

were used in the proof. Using these things only it is well known that we can prove that when $H = T$, the number of zeros is $\leq T^{\frac{3}{2}-\sigma}(\log T)^C$.

We now resume the Remark 1. We will show that (5) implies that there are points $t_1, t_2, \cdots, t_N$ (where $N \gg T$) with $|t_j - t_{j'}|$ bounded below whenever $j \neq j'$ and further

$$|F(\frac{1}{2} - \delta + it_j)| > T^{\frac{\delta}{10}}, \ (T \leq t_j \leq 2T). \tag{6}$$

After this we have only to apply Theorem 3 of § 4 in $[\mathrm{BR}]_3$ to obtain the proof of Conjecture 3. To deduce (6) from (5) we write

$$\psi(T) = \frac{1}{T}\int_T^{2T} |F(\frac{1}{2} - \delta + it)|\, dt. \tag{7}$$

Now RHS of (7) is

$$\frac{1}{T}\sum_I \int_I |F(\frac{1}{2} - \delta + it)|\, dt \tag{8}$$

where $\{I\}$ is a division of $[T, 2T]$ into disjoint (but abutting intervals I of equal length the length being both above and below). Plainly the expression (8) is

$$\leq \frac{2}{T}\sum_I{}' \int_I |F(\frac{1}{2} - \delta + it)|\, dt \tag{9}$$

where the accent denotes the restriction of the sum to intervals I for which the integral is not less than $\varepsilon\ \psi(T)$ for a small constant $\varepsilon > 0$. Now

$$\begin{aligned}
\psi(T) &\leq \tfrac{2}{T}\sum_I{}'\int_I |F(\frac{1}{2} - \delta + it)|\, dt \\
&\leq \tfrac{2}{T}\left(\sum_I{}' 1\right)^{\frac{1}{2}}\left(\sum_I{}'\left(\int_I |F(\frac{1}{2} - \delta + it)|\, dt\right)^2\right)^{\frac{1}{2}} \\
&\ll \tfrac{1}{T}\left(\sum_I{}' 1\right)^{\frac{1}{2}}\left(\sum_I{}'\int_I |F(\frac{1}{2} - \delta + it)|^2\, dt\right)^{\frac{1}{2}}.
\end{aligned}$$