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MODULAR FORMS AND ALGEBRAIC K-THEORY

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In this paper, which follows closely the talk given at the conference, I will sketch an example of a non-trivial element of K_2 of a certain threefold, whose existence is related to the vanishing of an *incomplete* L-function of a modular form at $s = 1$. To explain how this fits into a general picture, we begin with a simple account, for the non-specialist, of some of the conjectures (mostly due to Beilinson) which relate ranks of K-groups and orders of L-functions, supplemented by examples coming from modular forms. The picture presented is in some respects wildly distorted; among the important topics which are given little mention are:

(i) the connection between special values of L-functions and higher regulators, which is at the heart of the Beilinson conjectures;

(ii) the conjectures of Birch and Swinnerton-Dyer, and their generalisation by Beilinson and Bloch;

(iii) the theory of (mixed) motives, which underlies the constructions of the last section.

But I hope that it may be of some use as a gentle introduction to the subject, and to prepare the reader for a more comprehensive account (see for example **[9,17,18,21]** and above all **[1]**).

1. Beginnings

The story begins with Dirichlet's unit theorem: if F is a number field with ring of integers $\mathfrak{o}_F$, then

$$\operatorname{rk} \mathfrak{o}_F^* = r_1 + r_2 - 1 = \operatorname{ord}_{s=0} \zeta_F(s)$$

and there is the analytic class number formula, which at $s = 0$ reads:

$$\zeta_F^*(0) = -\frac{h_F R_F}{w_F} \tag{1}$$

where $\zeta_F^*(0)$ denotes the leading coefficient in the Taylor series of $\zeta_F(s)$ at $s = 0$. More generally, let S be a finite set of primes of F, and $\mathfrak{o}_{F,S}$ the ring of S-integers of F. Then the S-unit theorem says

$$\begin{aligned}\operatorname{rk} \mathfrak{o}_{F,S}^* &= r_1 + r_2 - 1 + \#S \\ &= \operatorname{ord}_{s=0} \zeta_{F,S}(s)\end{aligned}$$

where $\zeta_{F,S}(s)$ is the *incomplete* zeta function:

$$\zeta_{F,S}(s) = \prod_{\mathfrak{p} \notin S} (1 - N\mathfrak{p}^{-s})^{-1}.$$

and the analogue of (1) is the S-class number formula.

Borel found a generalisation of these results to the zeta function at arbitrary negative integers:

Theorem. [5] *Let $l > 0$ be an integer. Then $K_{2l}\mathfrak{o}_F$ is finite, and*

$$\begin{aligned}\operatorname{rk} K_{2l+1}\mathfrak{o}_F &= \begin{cases} r_1 + r_2 & l \text{ even} \\ r_2 & l \text{ odd} \end{cases} \\ &= \operatorname{ord}_{s=-l} \zeta_F(s).\end{aligned}$$

Moreover the leading coefficient $\zeta_F^*(-l)$ is equal, up to a non-zero rational factor, to a "higher regulator".

Remarks: (i) Here $K_i\mathfrak{o}_F$ are the higher K-groups of F, as defined by Quillen (see section 2). This is a natural generalisation of the unit theorem since $K_1\mathfrak{o}_F = \mathfrak{o}_F^*$. The fact that $K_i\mathfrak{o}_F$ are finitely generated was proved by Quillen.

(ii) The higher regulator is the determinant of a certain natural homomorphism

$$K_{2l+1}\mathfrak{o}_F \otimes \mathbf{R} \to \mathbf{R}^{m_l}, \quad m_l = \operatorname{ord}_{s=-l} \zeta_F(s).$$

(iii) The analogue of the S-unit theorem for these higher K-groups is uninteresting; on the one hand, one has

$$K_q\mathfrak{o}_{F,S} \otimes \mathbf{Q} = K_q\mathfrak{o}_F \otimes \mathbf{Q} = K_qF \otimes \mathbf{Q} \tag{2}$$

for every $q > 1$ (cf. section 2); on the other, the individual Euler factors in $\zeta_F(s)$ have no poles at negative integer points, so

$$\operatorname{ord}_{s=-l} \zeta_F(s) = \operatorname{ord}_{s=-l} \zeta_{F,S}(s)$$

for any finite set S of primes and any $l > 0$.

2. K-THEORY

For any scheme X there is a Grothendieck group K_0X. It is defined as the abelian group generated by symbols $[\mathcal{E}]$, where $\mathcal{E}$ runs over all isomorphism classes of vector bundles on X, with relations of the form

$$[\mathcal{E}] = [\mathcal{E}'] + [\mathcal{E}'']$$

for every exact sequence $0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$. For a ring R one can define K_0R to be $K_0 \operatorname{Spec} R$, or (which amounts to the same thing) as the Grothendieck group of projective R-modules, with relations $[M \oplus N] = [M] + [N]$.

In a similar way one also has the group $K_0'X$, generated by $[\mathcal{E}]$ for arbitrary coherent sheaves $\mathcal{E}$, with relations from exact sequences of coherent sheaves.

Quillen showed that K_0X and $K_0'X$ are part of an infinite sequence of groups K_qX, $K_q'X$ for $q \geq 0$, constructed as the higher homotopy groups π_{q+1} of certain spaces attached to X. For some of the different ways to define them, see **[10,16,22]**.

Among the important properties of these groups are:

(i) There are cup-products $K_pX \times K_qX \longrightarrow K_{p+q}X$;
(ii) For X regular (e.g. a smooth variety) $K_q'X = K_qX$;
(iii) For $Y \subset X$ a closed subscheme, there is a long exact sequence (the localisation sequence)

$$\cdots \longrightarrow K_q'Y \longrightarrow K_q'X \longrightarrow K_q'(X - Y) \longrightarrow K_{q-1}'Y \longrightarrow \ldots$$

(iv) $\mathcal{O}^*(X)$ injects into K_1X, with equality if $X = \operatorname{Spec} F$ is the spectrum of a field.
(v) The K-groups of finite fields are finite (of known order).

For a number field F the localisation sequence gives

$$\cdots \longrightarrow K_q\mathfrak{o}_F \longrightarrow K_q\mathfrak{o}_{F,S} \longrightarrow \coprod_{\mathfrak{p} \in S} K_{q-1}\mathfrak{o}_F/\mathfrak{p} \longrightarrow K_{q-1}\mathfrak{o}_F \longrightarrow \ldots$$

which together with (v) gives (2).

3. L-FUNCTIONS OF AN ALGEBRAIC VARIETY

Consider a smooth, projective algebraic variety X over $\mathbf{Q}$. Since any variety over a number field may be regarded—by restriction of scalars *à la* Grothendieck—as a variety over $\mathbf{Q}$ (in general, not geometrically connected) the restriction to ground field $\mathbf{Q}$ is not serious.

For each integer i in the range $0 \leq i \leq 2\dim X$ there is an L-function $L(h^i(X), s)$, which is an Euler product:

$$L(h^i(X), s) = \prod_p P_p^{(i)}(p^{-s})^{-1}.$$

The polynomials $P_p^{(i)}(t)$ here are defined as follows. Pick a prime $\ell \neq p$, and let $H^i_\ell(X)$ be the ℓ-adic cohomology of $X/\overline{\mathbf{Q}}$, which is a finite-dimensional $\mathbf{Q}_\ell$-vector space on which $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts continuously. Let $I_p \subset D_p \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ be inertia and decomposition subgroups at a prime of $\overline{\mathbf{Q}}$ over p, and $\mathrm{Frob}_p = \phi_p^{-1} \in D_p/I_p$ the inverse of the Frobenius substitution. Then

$$P_p^{(i)}(t) = \det\bigl(1 - t\,\mathrm{Frob}_p \bigm| H^i_\ell(X)^{I_p}\bigr)$$

is the characteristic polynomial of Frob_p (the "geometric Frobenius") acting on the inertia invariants.

If X has a good reduction X_p at p, then $P_p^{(i)}$ has integer coefficients, and does not depend on ℓ, by Deligne's proof of the Weil conjectures **[6]**; moreover in this case the zeroes of $P_p^{(i)}(t)$ all have absolute value $p^{-i/2}$. For general p it is conjectured that $P_p^{(i)}(t)$ has integer coefficients, is independent of ℓ, and that its roots have absolute values $p^{-j/2}$ for various integers $j \leq i$. This is known in very few cases (curves, a class of surfaces and some sporadic higher-dimensional examples). For the conjectures that follow to make sense, we must assume these local properties are true. It is then conjectured that $L(h^i(X), s)$—which is analytic and non-zero for $\Re(s) > i/2 + 1$, by the Euler product—has a meromorphic continuation satisfying a functional equation for the substitution $s \mapsto 1 + i - s$.

4. General conjectures

The part of Beilinson's conjecture related to orders of L-functions can now be approximately stated:

Let m be an integer satisfying $m \leq \frac{1+i}{2}$. Write $q = 1 + i - 2m$. Then the order of $L(h^i(X), s)$ at $s = m$ is equal to the dimension of a certain subspace of $K_q(X)_{\mathbf{Z}} \otimes \mathbf{Q}$. More precisely, for $q > 0$

$$\dim K_q X_{/\mathbf{Z}} \otimes \mathbf{Q} = \sum_{\substack{(i,m) \\ 1+i-2m=q}} \mathrm{ord}_{s=m} L(h^i(X), s).$$

Remarks: (i) The group $K_q X_{/\mathbf{Z}}$ is defined as follows. Let $\mathcal{X}$ be a regular model for X over $\mathbf{Z}$; in other words, $\mathcal{X}$ is a regular scheme, proper over $\mathrm{Spec}\,\mathbf{Z}$, such that $\mathcal{X} \otimes \mathbf{Q} = X$. Then

$$K_q X_{/\mathbf{Z}} = \mathrm{Image}\bigl(K_q \mathcal{X} \to K_q X\bigr).$$