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SOME REMARKS ON ELLIPTIC CURVES OVER FUNCTION FIELDS

Tetsuji SHIODA

In my lecture at the Journées Arithmétiques in Geneva (entitled "Mordell-Weil lattices and sphere packings"), I talked on
1) a brief survey on lattices and sphere packings,
2) basic results on Mordell-Weil lattices, and
3) application to sphere packings via supersingular surfaces.
For these topics, the following references are available: 1) [CS,Ch.1], 2) [S3], [S4] and 3) [E], [Oe], [S5].

In this note, instead of reporting on these, I would like to treat some related topics on elliptic curves over a function field, especially some results on the L-function of an elliptic curve over a function field with a finite constant field. Most of them must be known to experts, but the approach based on surface theory and Mordell-Weil lattices seems to provide a natural setting for this subject (cf. [T2],[G],[Mc]). In particular, this method enables one to write down explicit examples of such an L-function in some nontrivial cases.

The contents of this paper are as follows:
1. Elliptic surfaces
2. The L-function of an elliptic curve
3. Supersingular case
4. Rational elliptic surfaces

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1 Elliptic surfaces

Let us review first some basic results on elliptic surfaces, fixing the notation. Let k be an algebraically closed field of arbitrary characteristic and let K/k be a function field of one variable over k, i.e., $K = k(C)$ for some smooth projective curve C over k. Let E/K be an elliptic curve with a K-rational point O, and let $f : S \longrightarrow C$ denote the elliptic surface associated with E/K (the Kodaira-Néron model). The elliptic curve E is recovered from f as its generic fibre and, as is well known, the K-rational points of E can be identified with the sections of f; for each $P \in E(K)$, (P) denotes the image curve in S of the section $P : C \longrightarrow S$. We always assume the condition $(*)$ that f has at least one singular fibre.

Now let $N = \mathrm{NS}(S)$ be the Néron-Severi group of S; it is a free module of finite rank ρ (=the Picard number of S), which is an (indefinite) integral lattice with respect to the intersection pairing. We denote by T or L the trivial or essential sublattice of N; by definition, T is the sublattice generated by the zero-section (O), a fibre and all components of reducible fibres of f, and L is the orthogonal complement of T in N. In particular, we have

$$N \otimes \mathbf{Q} = (T \otimes \mathbf{Q}) \oplus (L \otimes \mathbf{Q}) \tag{1.1}$$

and

$$\rho = \mathrm{rk}\, T + \mathrm{rk}\, L. \tag{1.2}$$

Further we have

$$\mathrm{rk}\, T = 2 + \sum_{v \in C} (m_v - 1) \tag{1.3}$$

where m_v is the number of irreducible components of the fibre $f^{-1}(v)$, and $\mathrm{rk}\, L$ is equal to the Mordell-Weil rank of E/K:

$$r := \mathrm{rk}\, L = \mathrm{rk}\, E(K). \tag{1.4}$$

Actually there is a natural isomorphism

$$L \otimes \mathbf{Q} \simeq E(K) \otimes \mathbf{Q}, \tag{1.5}$$

which takes the intersection pairing on L to the height pairing on the Mordell-Weil group (up to the sign change); indeed this is essentially how we defined the structure of Mordell-Weil lattices (see [S4]).

Next we consider the cycle map

$$\gamma : N \longrightarrow H = H^2(S, \mathbf{Q}_l(1)) \tag{1.6}$$

where H stands for the l-adic cohomology group with a fixed prime number $l \neq \mathrm{char}(k)$ (cf. [T1]). It is injective and takes the intersection pairing of N into the cup-product pairing in H. Let us denote by Trans(S) the orthogonal complement of $\mathrm{Im}(\gamma)$ in H, whose elements are called transcendental cycles on S, and by W the orthogonal complement of $\gamma(T)$ in H. The space W corresponds to what Weil called the *essential part* in the second homology of S (cf. his comments to the paper [1967a] in [W, III]). Then we have

$$H \simeq (N \otimes \mathbf{Q}_l) \oplus \mathrm{Trans}(S) \simeq (T \otimes \mathbf{Q}_l) \oplus W \tag{1.7}$$

and

$$W \simeq (L \otimes \mathbf{Q}_l) \oplus \mathrm{Trans}(S) \tag{1.8}$$

The Lefschetz number of S is defined as

$$\lambda := \dim \mathrm{Trans}(S) = b_2 - \rho \qquad (b_2 = \dim H^2(S)) \tag{1.9}$$

which is known to be a birational invariant of S.

Proposition 1 *The dimension w of the vector space W is given by*

$$w = r + \lambda = b_2 - \mathrm{rk}\, T. \tag{1.10}$$

If $\mathrm{char}(k) \neq 2, 3$, *then*

$$w = 4g - 4 + \mu + 2\alpha \tag{1.11}$$

where g is the genus of C (or of K) and μ (resp. α) is the number of singular fibres of multiplicative (resp. additive) type.

Proof. The first part is immediate from (1.7) and (1.8). The second part is also well-known (cf.[R],[S1]). Let us briefly recall the idea of the proof. From the standard facts in surface theory, we have

$$b_2 = c_2 + 2b_1 - 2 \qquad (c_2 = \text{ Euler number of } S)$$

where $b_1 = 2g$ since we are assuming the condition $(*)$. On the other hand, we have the following formula for $\mathrm{char}(k) \neq 2, 3$:

$$c_2 = \sum_v e_v \qquad (e_v = \text{ Euler number of } f^{-1}(v)) \tag{1.12}$$

(cf. [K],[Ogg],[Ogu]). Then, by (1.9) and (1.3), we have

$$w = 4g - 4 + \sum_v (e_v - m_v + 1).$$

It remains to check that

$$e_v = m_v \text{ or } m_v + 1$$

according as the fibre $f^{-1}(v)$ is of multiplicative or additive type, which can be done using the classification of singular fibres([K],[N],[T3]). *q.e.d.*

It may be worthwhile to mention the following direct consequence. Simply note that we have $\lambda \geq 0$ in general and $\lambda \geq 2p_g$ (p_g: geometric genus of S) in characteristic 0.

Corollary 2 *If* char$(k) \neq 2,3$, *then*

(1.13). $$r \leq w = 4g - 4 + \mu + 2\alpha.$$

Corollary 3 *Assume* char$(k) = 0$. *Then*

(1.14) $$p_g \leq \frac{1}{2}w = 2g - 2 + \frac{1}{2}\mu + \alpha$$

(1.15) $$c_2 = 12(p_g - g + 1) \leq 6(2g - 2 + \mu + 2\alpha)$$

and

(1.13′). $$r \leq 4g - 4 + \mu + 2\alpha - 2p_g.$$

Remark. (a) In case char$(k) = 2$ or 3, (1.10) is still valid, but (1.11) should be modified by adding an extra term caused by wild ramifications (cf. [Ogg],[R],[Sa]). In other words, each e_v in (1.12) should be replaced by $e_v + \delta_v$ with a well-defined non-negative integer δ_v so that the right hand side of (1.11) should have the term $\sum_v \delta_v$.
(b) The idea behind equality of expressions in (1.10) and (1.11) was first used by Igusa [I] to define a correct Betti number b_2 of an algebraic surface, and later it was formulated in a more general situation as the so-called Ogg-Shafarevich formula (cf. [R]).
(c) The above (1.14) or its equivalent (1.15) seems to have been proved by many authors again and again, though it was explicitly stated in [S1,Cor.2.7] in 1972. In particular, (1.15) is sometimes called Szpiro's conjecture (cf. [Sz,p.10]); note that we make no assumption of semi-stability ($\alpha = 0$) in the above argument.