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1. Introduction

In this paper we describe a representation theorem for solutions of the differential equation

$$\Delta u + \lambda q(x) u = 0 \tag{1.1}$$

on certain non-compact real analytic Riemannian manifolds. Here Δ is the Laplace-Beltrami operator, λ a complex number and $q(x)$ is a positive real-analytic function. The theorem is a generalization of a representation theorem for solutions of the Helmholtz equation on hyperbolic space proved by Helgason [3; 4] and Minemura [5]. By way of introduction we recall this special representation theorem.

We take for the hyperbolic n-space the Poincaré model of the unit ball $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$ with the Riemannian metric

$$ds^2 = (\frac{1 - |x|^2}{2})^{-2} |dx|^2. \tag{1.2}$$

$\mathbb{B}^n$ is a complete non-compact Riemannian manifold with an ideal boundary $\partial\mathbb{B}^n$ identified with the sphere $S^{n-1} \subset \mathbb{R}^n$. The Laplace-Beltrami operator on $\mathbb{B}^n$, denoted by Δ_h, is given in Euclidean global coordinates by

$$\Delta_h = (\frac{1 - |x|^2}{2})^2 \Delta + (n-2)\frac{1 - |x|^2}{2} \sum_{i=1}^{n} x_i \partial_i \tag{1.3}$$

where Δ is the Laplacian on $\mathbb{R}^n$, $\partial_i = \partial/\partial x_i$.

Consider the equation

$$\Delta_h u + \lambda u = 0 \text{ in } \mathbb{B}^n. \tag{1.4}$$

The Helmholtz equation (1.4) has a distinguished class of solutions known as the generalized eigenfunctions of $-\Delta_h$. Given any $s \in \mathbb{C}$ and $\omega \in \partial\mathbb{B}^n$ there is a unique (normalized) generalized eigenfunction denoted by $E(x, \omega; s)$, $x \in \mathbb{B}^n$. In Euclidean coordinates it has the explicit form

$$E(x, \omega; s) = (\frac{1 - |x|^2}{|x - \omega|})^s \tag{1.5}$$

for $|x| < 1$, $\omega \in S^{n-1}$. The function $u(x) = E(x, \omega; s)$ is a solution of equation (1.4) with $\lambda = s(n-1-s)$. The problem arises whether any solution u of equation (1.4) can be represented by an integral formula of the form

$$u(x) = \int_{S^{n-1}} \Phi(\omega) E(x, \omega; s) d\omega,$$

for s satisfying $s(n-1-s) = \lambda$, where Φ is some generalized function on S^{n-1}. This problem was solved in the affirmative by Helgason [3;4] and by Minemura [5]. Their main result can be stated as follows,

THEOREM 1.1. *Let $u(x)$ be a solution of the Helmholtz equation*

$$\Delta_h u + s(n-1-s)u = 0 \text{ in } \mathbb{B}^n \tag{1.6}$$

where s is some complex number such that $s \neq (n-1-j)/2$ for $j = 1, 2, \ldots$ Then there exists a unique hyperfunction Φ_u on S^{n-1} such that

$$u(x) = \langle \Phi_u, E(x, \cdot; s) \rangle \tag{1.7}$$

for $x \in \mathbb{B}^n$. Moreover, the map: $u \to \Phi_u$ is a bijection of the space of solutions of (1.6) on the space of hyperfunctions on S^{n-1}.

In this paper we generalize Theorem 1.1 and show that a similar representation theorem holds for solutions of equations (1.1) on a general class of non-compact Riemannian manifolds of which hyperbolic space is a special case. We use a P.D.E. oriented approach. When restricted to the special situation of Theorem 1.1 our approach yields a new proof of the theorem which is not using the special structure of $\mathbb{B}^n$ as a symmetric space (see also [1]). The general set up of our study is as follows. Let X be a real-analytic compact Riemannian manifold with a boundary ∂X. Let g denote the Riemannian metric on X and let Δ_g denote the corresponding Laplace-Beltrami operator. Set

$$\overset{\circ}{X} = X \setminus \partial X.$$

Introduce on $\overset{\circ}{X}$ a new Riemannian metric h, conformal with g, defined by

$$h = \rho^{-2} g \tag{1.8}$$

where $\rho(x)$ is a real-analytic function on X such that

$$\begin{aligned} &\rho(x) > 0 \text{ on } \overset{\circ}{X}, \\ &\rho(x) = 0 \text{ and } d\rho \neq 0 \text{ on } \partial X. \end{aligned} \tag{1.9}$$

Denote by Δ_h the Laplace-Beltrami operator on $\overset{\circ}{X}$ in the metric h. It is given by

$$\Delta_h = \rho^2 \Delta_g - (n-2)\rho(\nabla_g \rho) \tag{1.10}$$

where throughout the paper n denotes the dimension of X and where $\nabla_g \rho$ denotes the gradient vector field in the metric g. As usual $\nabla_g \rho$ is identified with a first order differential operator given in local coordinates by

$$\nabla_g \rho = \sum_{i,j} g^{ij} \frac{\partial \rho}{\partial x_i} \frac{\partial}{\partial x_j}. \tag{1.11}$$

We denote by $|\nabla_g \rho(x)|$, the norm of the vector $\nabla_g \rho(x)$ induced by g. In local coordinates

$$|\nabla_g \rho(x)|^2 = \sum_{i,j} g^{ij}(x) \frac{\partial \rho}{\partial x_i} \frac{\partial \rho}{\partial x_j}. \tag{1.12}$$

We consider solutions of the differential equation

$$\Delta_h u + \lambda q(x) u = 0 \text{ in } \overset{\circ}{X} \tag{1.13}$$

where $q(x)$ is a positive real-analytic function on X and λ is a complex number. We shall derive a representation theorem similar to Theorem 1.1 for solutions of (1.13). It will involve the generalized eigenfunctions of the operator $q^{-1}\Delta_h$ which will be defined in section 2.

REMARK: We note that the main result of this paper (the representation theorem) holds under weaker smoothness assumptions than those imposed above. The result holds if one assumes for instance that X, ρ and q are of class C^2 and that in addition X, ρ and q are real analytic in some neighborhood of ∂X.

In this paper we are going to impose on the function q a boundary condition. We shall assume that

$$q(x) = |\nabla_g \rho(x)|^2 \text{ on } \partial X. \tag{1.14}$$

We note that this condition is not necessary for the validity of the main representation theorem. However assumption (1.14) simplifies considerably many details in the proof of the theorem. Observe that equation (1.6) on hyperbolic n-space belongs to the class of equations introduced above. We conclude this introduction by noting that the representation theorem described in this paper for solutions of (1.13) can be shown to hold for solutions of a much wider class of equations of the form

$$\rho^2 \Delta_g u + \rho B u + C u = 0 \text{ in } \overset{\circ}{X}$$

where B is a real-analytic vector field on X satisfying some conditions on ∂X and C is a real-analytic function on X.

The main part of this paper is divided into two sections. In section 2 we discuss the Green's function associated with equation (1.13). The asymptotic and related real-analyticity properties of the Green's function play a crucial role in our study. These are described in Theorem 2.1. Using the theorem we define the generalized eigenfunctions which form a distinguished class of solutions of equation (1.13) and which are the building blocks in the representation theorem for any solution of that equation. The representation theorem is stated and proved in section 3. We note that the proof of the theorem