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SINGULAR PERTURBATIONS OF DIRICHLET AND NEUMANN DOMAINS AND RESONANCES FOR OBSTACLE SCATTERING

Peter D. Hislop [1]

1. Introduction

Some of the work reported in this article is joint with R.M. Brown, University of Kentucky, and A. Martinez, Université de Paris XIII. We want to describe some recent results concerning the existence and estimation of the poles of the S-matrix for the scattering of waves by a single, compact obstacle. The details of the calculations appear in [6], [12], [11]. We are interested in the scattering poles for a class of obstacles known as Helmholtz resonators. These obstacles are characterized by a large cavity $\mathcal{C}$ which is coupled to the (unbounded) exterior $\mathcal{E}$ by means of a tube $T(\varepsilon)$ of diameter ε. The waves propagate in $\Omega(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon) \cup \mathcal{E}})$ and we consider either Dirichlet or Neumann boundary conditions (DBC or NBC) on the boundary of $\Omega(\varepsilon), \partial\Omega(\varepsilon)$. We consider two classes of problems : (1) local in energy : for a fixed compact subset $K \subset \mathbf{C}$, intersecting the real axis $\mathbf{R}$, describe and estimate the position of all scattering poles in K for all ε sufficiently small; (2) global in energy : for a fixed ε (say $\varepsilon = 1$), consider the high energy behavior of the scattering poles and show that there exists a sequence of poles converging to the real axis.

The problem of a local characterization of scattering poles for a Helmholtz resonator has been considered by Beale [4] and Arsen'ev [3]. For the case of DBC, the poles arise from either eigenvalues of the cavity Laplacian $-\Delta_{\mathcal{C}}$ with DBC or resonances of the exterior Laplacian $-\Delta_{\mathcal{E}}$ with DBC. In particular for K as above, they prove that there exists $\varepsilon_K > 0$ such that for all $\varepsilon < \varepsilon_K$, there exists a bijection between the scattering poles in K and the set consisting of the eigenvalues of $-\Delta_{\mathcal{C}}$ in K and the resonances of $-\Delta_{\mathcal{E}}$ in K (including multiplicities).

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When there are NBC and $T(\varepsilon)$ is a straight tube $D_\varepsilon \times [0,1], D_\varepsilon = \varepsilon D_1$ and $D_1 = \{x' \epsilon \mathbf{R}^{n-1} \mid |x'| \leq 1\}$, there is an addition set of poles coming from the longitudinal modes of the tube. We reprove these results and give precise upper bounds on the displacement of the poles from the cavity eigenvalues or exterior resonances as a function of ε. For the case of DBC these are exponentially small in ε. For the NBC case, the upper bound is $\mathcal{O}(\varepsilon^\beta)$ where $\beta = 1/2$ for dimension $n \geq 4$ and $0 < \beta < 1/2$ for $n = 3$.

In order to derive these results, we also study the effect of adding a small tube $T(\varepsilon)$ to the cavity $\mathcal{C}$ on the eigenvalues of $-\Delta_{\mathcal{C}}$. We consider both DBC and NBC. In the DBC case, we find that the shift of the eigenvalues is bounded above by $\mathcal{O}(\varepsilon^\beta)$ where $\beta = 1/2$ for $n \geq 3$ and $0 < \beta < 1/2$ for $n = 2$.

In the NBC case, we must restrict ourselves to a straight tube. We find a similar estimate for the shift of the eigenvalues. We mention that singular perturbations of NBC have been recently discussed by several authors, for example [2], [10], [16].

The second type of problem is related to a conjecture of Lax and Phillips [18] concerning the behavior of scattering poles in the case that the obstacle has trapped rays. They conjectured that if an obstacle, like a Helmholtz resonator, has trapped rays, then there is a sequence of scattering poles converging to the real axis as the energy diverges to infinity. Although this conjecture is false, as shown by Ikawa [13] for the case of two bounded, convex obstacles with a single trapped hyperbolic ray, we show that it holds for a class of symmetric Helmholtz resonators (see section 4). In the case studied by Ikawa and, later, by Gérard [9], there is an infinite number of scattering poles but they are bounded a fixed distance from the real axis. This may be a manifestation of the instability of the trapped ray in this example. Indeed, Ikawa [14] later showed that if the obstacles are sufficiently flat in the neighborhood of the trapped ray, there is a sequence of poles converging to the real axis. A similar situation of stability occurs in an example studied by Ralston [20]. He examined the poles for scattering in spherically symmetric inhomogeneous media for which there is an infinite family of stable, trapped rays. Again in this case, there is a sequence of poles converging exponentially fast to the real axis. This model can also be treated by the methods of section 4.

The outline of this paper is as follows. In sections 2 and 3 we discuss the local in energy problem for the Helmholtz resonator. Section 2 is devoted to the DBC case and section 3 to the NBC case. In section 4 we turn to the global in energy problem and sketch the proof of the Lax-Phillips conjecture on the existence of a sequence of scattering poles converging to the real axis for a family of symmetric Helmholtz resonators.

Finally, we mention that a scattering pole is also a pole of the meromorphic continuation of matrix elements of the resolvent of $-\Delta_{\Omega(\varepsilon)}$ for vectors in a certain dense set. Hence they are resonance of the operator $-\Delta_{\Omega(\varepsilon)}$ on $L^2(\Omega(\varepsilon))$. We will freely use the results of the theory of quantum resonances and spectral deformation below. In particular, we will assume the application of spectral deformation techniques as discussed in [12].

2. Perturbation of Dirichlet Domains and Resonances

The first situation for which we will consider the local resonance structure is the Helmholtz resonator with DBC. This material has already been published so we will be brief and simply review the results. The notation and general ideas, however, will be used in the other sections. To be more specific about the geometry, let $\widetilde{\Omega} \subset \mathbf{R}^n$ be an open set with C^2-boundary admitting a decomposition into two disjoint components $\mathcal{C}$, the cavity, and $\mathcal{E}$, the exterior, such that $\mathcal{C} \subset \mathbf{R}^n \backslash \mathcal{E}$ and $\mathcal{C}$ is bounded. Let $x_0 \epsilon \partial \mathcal{C}$ and $x_1 \epsilon \partial \mathcal{E}$. We join these two points by a tube $T(\varepsilon)$ which is an open subset of $\mathbf{R}^n \backslash \widetilde{\Omega}$ diffeomorphic to the standard tube $D_\varepsilon \times [0,1]$ where $D_\varepsilon = \varepsilon D_1$ and $D_1 \equiv \{x' \epsilon \mathbf{R}^{n-1} \mid |x'| \leq 1\}$. As in the introduction, we set $\Omega(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon) \cup \mathcal{E}})$ and consider the Laplacian $H_\varepsilon = -\Delta$ on $\Omega(\varepsilon)$ with DBC on $\partial\Omega(\varepsilon)$. Our main result is to characterize the resonances of H_ε in a compact complex set K intersecting $\mathbf{R}$ for all ε sufficiently small.

To this end, we need a preliminary estimate of some interest in itself. Consider the cavity $\mathcal{C}$ and the cavity with the tube $T(\varepsilon)$ attached : $\mathcal{C}(\varepsilon) \equiv Int(\overline{\mathcal{C} \cup T(\varepsilon)})$, both with DBC. We want to know by how much the eigenvalues of the Dirichlet Laplacian $-\Delta_{\mathcal{C}}$ shift when the tube is adjoined to the cavity. By the Poincaré inequality for $-\Delta_{T(\varepsilon)}$, one expects that the effect is small.

PROPOSITION **2.1.** *Let $\lambda_0 \epsilon\ \sigma(-\Delta_{\mathcal{C}})$ with multiplicity N_0. Then there exists $\varepsilon_0 > 0, c > 0$ such that for all $\varepsilon < \varepsilon_0, -\Delta_{\mathcal{C}(\varepsilon)}$ has N_0 eigenvalues (counting multiplicity) $\lambda_1(\varepsilon), \ldots, \lambda_{N_0}(\varepsilon)$, satisfying for all $j = 1, \ldots, N_0$:*

$$|\lambda_0 - \lambda_j(\varepsilon)| \leq c\varepsilon^\beta$$

where $\beta = 1/2$ for $n \geq 3$ and $0 < \beta < 1/2$ for $n = 2$.

The proof of this theorem begins with Green's formula expressing the difference of the two Laplacians, $-\Delta_{\mathcal{C}} \oplus -\Delta_{T(\varepsilon)}$ and $-\Delta_{\mathcal{C}(\varepsilon)}$, in terms of normal derivatives and surface integrals. These integrals are then estimated using Sobolev embedding and trace theorems.

The basis for the existence of resonances in K is the fact that a narrow tube with Dirichlet boundary conditions cannot support states with energy in K if ε is sufficiently small. Consequently, the coupling between the cavity and the exterior is very weak. This weak coupling, however, is sufficient to change the bound states of $-\Delta_{\mathcal{C}}$ to resonances of H_ε and to shift the resonances of $-\Delta_{\mathcal{E}}$ a small amount to become resonances of H_ε. We note that $\sigma(H_\varepsilon) = [0, \infty)$ and is absolutely continuous whereas the spectrum of the operator obtained when $\varepsilon = 0$, a direct sum, has eigenvalues embedded in the continuous spectrum.

As described in the introduction, the poles of the S-matrix are characterized also as the complex eigenvalues of the spectrally deformed Hamiltonian. We denote by $H_\varepsilon(\mu), H_\varepsilon^{ext}(\mu)$ and $H_{ext,\varepsilon}^D(\mu)$ the spectrally deformed operators obtained from $H_\varepsilon, -\Delta_{\mathcal{E}(\varepsilon)}$ and $-\Delta_{T(\varepsilon)} \oplus -\Delta_{\mathcal{E}}$, respectively, where $\mathcal{E}(\varepsilon) \equiv Int(\overline{\mathcal{E} \cup T(\varepsilon)})$. There is a result for the shift of the resonances of $-\Delta_{\mathcal{E}}$ by the addition of $T(\varepsilon)$, which is the analog of Proposition 2.1.

PROPOSITION **2.2.** *Let λ_0 be a resonance of $-\Delta_{\mathcal{E}}$ for some $\mu \epsilon\ i]0,1[$ of (algebraic) multiplicity N_0. Then there exists $\varepsilon_0 > 0, c > 0$ such that for all $\varepsilon < \varepsilon_0, H_\varepsilon^{ext}(\mu)$ has N_0 eigenvalues $\lambda_1(\varepsilon), \ldots, \lambda_{N_0}(\varepsilon)$ satisfying for all $j = 1, \ldots, N_0$:*

$$|\lambda_0 - \lambda_j(\varepsilon)| \leq c\varepsilon^\beta$$

where $\beta = 1/2$ for $n \geq 3$ and $0 < \beta < 1/2$ for $n = 2$.

To prove that H_ε has resonances in some fixed $K \subset \mathbf{C}$, for all small ε, and that these resonances are precisely, those coming from the eigenvalues of $-\Delta_{\mathcal{C}}$ in K and the resonances of $-\Delta_{\mathcal{E}}$ in K, we show that for z in a neighborhood of any of these latter points, the difference of the resolvents of $H_{\varepsilon,\mu}$ and of $-\Delta_{\mathcal{C}(\varepsilon)} \oplus H_\varepsilon^{ext}(\mu)$, $\mu \epsilon\ i]0,1[$ vanishes as $\varepsilon \to 0$. Note that $\mathcal{C}(\varepsilon) \cap \mathcal{E}(\varepsilon) = T(\varepsilon)$ and it is in this region where states of energy in K are, in fact, exponentially small (see below). To quantify this idea, we use geometric perturbation theory. Let (J_1, J_2) be a partition of unity covering $\Omega(\varepsilon)$, independent of ε, such that $\text{supp}|\nabla J_i|$ is well inside the tube. Indeed, if $d(x, \Omega) \equiv$ Euclidean distance from x to Ω, then we take

$$J_1 | \{x | d(x, \mathcal{E}) \geq 2\delta\} = 1$$

$$J_2 | \{x | d(x, \mathcal{E}) \geq \delta\} = 1$$

so $\text{supp}|\nabla J_i| \subset \{x | \delta \leq d(x, \mathcal{E}) \leq 2\delta\}$. Set $\mathcal{H}_0 \equiv L^2(\mathcal{C}(\varepsilon)) \oplus L^2(\mathcal{E}(\varepsilon))$ and $\mathcal{H} \equiv L^2(\Omega(\varepsilon))$ and define $J : \mathcal{H} \to \mathcal{H}_0$ by

$$Ju = J_1 u \oplus J_2 u$$

so that $J^\star J = 1_{\mathcal{H}}$. Let $R(z) \equiv (H_{\varepsilon,\mu} - z)^{-1}$ and $R_0(z) = (-\Delta_{\mathcal{C}(\varepsilon)} \oplus H_\varepsilon^{ext}(\mu) - z)^{-1}$. Then for z in the intersection of the resolvent sets, we have the geometric resolvent equation on $\mathcal{H}$:

$$R(z) = J^\star R_0(z) J + R(z) M R_0(z) J$$