Astérisque

MITSURU IKAWA

Singular perturbation of symbolic flows and the modified Lax-Phillips conjecture

Astérisque, tome 210 (1992), p. 217-235
<http://www.numdam.org/item?id=AST_1992__210__217_0>

SINGULAR PERTURBATION OF SYMBOLIC FLOWS AND THE MODIFIED LAX-PHILLIPS CONJECTURE

MITSURU IKAWA

1. Introduction. In the study of scattering by an obstacle consisting of several convex bodies, it is known that the distribution of poles of the scattering matrix has a close relationship to the zeta functions associated with a dynamical system in the exterior of the obstacle. When we want to consider the validity of the modified Lax-Phillips conjecture, we can derive it from the existence of poles of the zeta functions. That is, roughly speaking, if the zeta function has a pole in a certain region, the scattering matrix for the obstacle has an infinite number of poles in a strip $\{z \in \mathbf{C}; 0 < \operatorname{Im} z < \alpha\}$ for some $\alpha > 0$. The modified Lax-Phillips conjectrue will be explained in the next section.

Therefore, in order to consider distributions of poles of scattering matrices for an obstacle consisting of several convex bodies, the zeta functions play a crucial role. But unfortunately, it is not so easy to show the existence of a pole of the zeta functions in general.

In this talk, we shall develop a theory of singular perturbations of symbolic dynamics, with which we shall show the existence of a pole of the zeta function when the obstacle is consisted of several small balls.

In Section 2, we explain the modified Lax-Phillips conjecture and consider the scattering by obstacles consisting of several convex bodies. In Section 3, we shall discuss singular perturbations of symbolic dynamics. In Section 4, we shall show how to apply the theorem on singular perturbations of symbolic dynamics to considerations of the matrices for obstacles consisting of several small balls.

2. Scattering by several convex bodies.

Let $\mathcal{O}$ be a bounded open set in $\mathbf{R}^3$ with smooth boundary Γ. We set

$$\Omega = \mathbf{R}^3 - \overline{\mathcal{O}},$$

and assume that Ω is connected. Consider the following acoustic problem:

$$(2.1) \qquad \begin{cases} \Box u = \dfrac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{in } \Omega \times (-\infty, \infty), \\ u = 0 & \text{on } \Gamma \times (-\infty, \infty), \\ u(x,0) = f_1(x), \dfrac{\partial u}{\partial t}(x,0) = f_2(x). \end{cases}$$

We denote by $\mathcal{S}(z)$ the scattering matrix for this problem. The scattering matrix $\mathcal{S}(z)$ is an $\mathcal{L}(L^2(S^2))$-valued function analytic in $\{z; \operatorname{Im} z \leq 0\}$ and meromorphic in the whole complex plane $\mathbf{C}$, and that the correspondance from obstacles to scattering matrices

$$\mathcal{O} \to \mathcal{S}(z)$$

is one to one(see for example [LP]).

Concerning the above correspondance, we are interested in the problem to know how the distribution of poles of scattering matrices relates to the geometry of obstacles. As to this problem, we would like to present the following conjecture:

Modified Lax-Phillips Conjecture. *When $\mathcal{O}$ is trapping, there is a positive constant α such that the scattering matrix $\mathcal{S}(z)$ has an infinite number of poles in $\{z; 0 < Im\, z \leq \alpha\}$.*

Hereafter, we say that MLPC(abbreviation of the modified Lax-Phillips conjecture) is valid for obstacle $\mathcal{O}$, when there is $\alpha > 0$ such that the scattering matrix $\mathcal{S}(z)$ corresponding to $\mathcal{O}$ has an infinite number of poles in $\{z; \operatorname{Im} z \leq \alpha\}$.

About this conjecture, obstacles consisting of two convex bodies were studied first. By the works [BGR], [G], [Ik1] and [S], the distribution of poles are well studied, and it is shown that MLPC is valid for obstacles consisting of two convex bodies. It is very natural to proceed to obstacles consisting of three strictly convex bodies. But the problem for three bodies exposes an essential difference from that of two bodies. Namely, for an obstacle consisting of three bodies, there exist infinitely many primitive periodic rays in the exterior of the obstacle in general. Thus, we have to consider geometric property of the totality of the periodic rays in the exterior, and it seems that the asymptotic behavior of the periodic rays with very large period plays an essential role.

Here, we present a theorem in [Ik3,4], which allows us to connect the asymptotic behavior of the periodic rays and the distribution of poles of the scattering matrix.

Let $\mathcal{O}_j,\ j=1,2,\cdots,L$, be bounded open sets with smooth boundary Γ_j satisfying

$$\text{(H.1)} \qquad \text{every } \mathcal{O}_j \text{ is strictly convex,}$$

$$\text{(H.2)} \quad \text{for every } \{j_1,j_2,j_3\} \in \{1,2,\cdots,L\}^3 \text{ such that } j_l \neq j_{l'} \quad \text{if} \quad l \neq l',$$

$$(\text{convex hull of } \overline{\mathcal{O}_{j_1}} \text{and } \overline{\mathcal{O}_{j_2}}) \cap \overline{\mathcal{O}_{j_3}} = \phi.$$

We set

$$\mathcal{O} = \cup_{j=1}^{L} \mathcal{O}_j, \quad \Omega = \mathbf{R}^3 - \overline{\mathcal{O}} \ \text{ and } \ \Gamma = \partial\Omega. \tag{2.2}$$

Denote by γ an oriented periodic ray in Ω, and we shall use the following notations:

$$\begin{aligned} d_\gamma &: \text{ the length of } \gamma, \\ T_\gamma &: \text{ the primitive period of } \gamma, \\ i_\gamma &: \text{ the number of the reflecting points of } \gamma, \\ P_\gamma &: \text{ the Poincaré map of } \gamma. \end{aligned}$$

We define a function $F_D(s)$ $(s \in \mathbf{C})$ by

$$F_D(s) = \sum_{\gamma} (-1)^{i_\gamma} T_\gamma |I - P_\gamma|^{-1/2} e^{-sd_\gamma} \tag{2.3}$$

where the summation is taken over all the oriented periodic rays in Ω and $|I - P_\gamma|$ denotes the determinant of $I - P_\gamma$.

Concerning the periodic rays in Ω we have

$$\#\{\gamma; \text{periodic ray in } \Omega \text{ such that } d_\gamma < r\} < e^{a_0 r} \tag{2.4}$$

and

$$|I - P_\gamma| \geq e^{2a_1 d_\gamma}, \tag{2.5}$$

where a_0 and a_1 are positive constants depending on $\mathcal{O}$. The estimates (2.4) and (2.5) imply that the right hand side of (2.3) converges absolutely in $\{s \in \mathbf{C};\ \mathrm{Re}\, s > a_0 - a_1\}$. Thus $F_D(s)$ is well defined in $\{s \in \mathbf{C};\ \mathrm{Re}\, s > a_0 - a_1\}$, and holomorphic in this domain.

Now we have

Theorem 2.1. *Let $\mathcal{O}$ be an obstacle given by (2.2) satisfying (H.1) and (H.2). If $F_D(s)$ cannot be prolonged analytically to an entire function, then MLPC is valid for $\mathcal{O}$.*

We cannot give here the proof of the above theorem. We would like to refer that the trace formula due to [BGR] is the starting point of the proof. This trace formula is written as follows:

$$\text{Trace}_{L^2(\mathbf{R}^3)} \int \rho(t) \left(\cos t\sqrt{-A} \oplus 0 - \cos t\sqrt{-A_0}\right) dt = \frac{1}{2} \sum_{j=1}^{\infty} \hat{\rho}(z_j), \qquad \text{for all } \rho \in C_0^\infty(0,\infty) \tag{2.6}$$

where

$$\hat{\rho}(z) = \int e^{izt} \rho(t) dt,$$

$\{z_j\}_{j=1}^\infty$ is a numbering of all the poles of $\mathcal{S}(z)$, A is the selfajoint realization in $L^2(\Omega)$ of the Laplacian with the Dirichlet boundary condition and A_0 the one in $L^2(\mathbf{R}^3)$, and $\oplus 0$ indicates the extension into $\mathcal{O}$ by 0. It gives us an relationship between the distribution of poles of the scattering matrix and the singularities of the trace of the evolution operator of (2.1). We shall use (2.6) in the following way: Suppose that $F_D(s)$ has a singularity. This enable us to choose a sequence of ρ of the form

$$\rho_q(t) = \rho(m_q(t - l_q))$$

in such way that

$$l_q \to \infty, \quad m_q \to \infty \quad \text{as } q \to \infty,$$

and that the left hand side does not decay so fast as q tends to the infinity. But if MLPC is not valid, the right hand side of (2.6) for ρ_q decreases very rapidely. The difference in decreasing speeds brings a contradiction. Thus MLPC is valid. The detailed proof is given in [Ik3].

By virtue of Theorem 2.1, the proof of the validity of MLPC is transfered to the consideration of singularities of $F_D(s)$. But it is not easy to show the existence of singularities of $F_D(s)$ in general. At present we can show it only for obstacles consisting of small balls.

Theorem 2.2. *Let $P_j, j = 1, 2, \cdots, L$, be points in $\mathbf{R}^3$, and set for $\varepsilon > 0$*

$$\mathcal{O}_\varepsilon = \cup_{j=1}^L \mathcal{O}_{j,\varepsilon}, \quad \mathcal{O}_{j,\varepsilon} = \{x; |x - P_j| < \varepsilon\}.$$