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Resolvent Estimates and Time-Decay in the Semiclassical Limit

SHU NAKAMURA

1. Introduction.

In this note we study the Schrödinger operator :

$$H = -(\hbar^2/2)\Delta + V(x), \qquad \text{on} \quad L^2(\mathbf{R}^d), \quad \hbar > 0$$

in the semiclassical limit: $\hbar \to 0$. In particular, we are interested in the scattering theory and long time behaviors of the time evolution: $e^{-itH/\hbar}\varphi$. Boundary value of the resolvent: $\lim_{\varepsilon\to+0}(H - \lambda \pm i\varepsilon)^{-1} = (H - \lambda \pm i0)^{-1}$ plays essential roles in the scattering theory, and various observable quantities, e.g., scattering amplitude, time-delay, etc., are represented by it ([RS]). In studying the boundary value of the resolvent, the theory of Mourre is quite powerful and has been applied to many problems (e.g., [M], [PSS], [CFKS]). Jensen, Mourre and Perry extended the theory using multiple commutators, and proved the existence of boundary values of *powers* of the resolvent ([JMP]). Using the result they also obtained time-decay results (see also [J1]).

In a series of papers [RT1]–[RT4], Robert and Tamura systematically studied the semiclassical limit of the scattering process for nontrapping energies. In their arguments, an estimate of the form:

$$\left\| \langle x\rangle^{-\alpha} (H - \lambda \pm i0)^{-1} \langle x\rangle^{-\alpha} \right\| \le C_\alpha \hbar^{-1}, \qquad \hbar > 0, \alpha > 1/2,$$

which is called *semiclassical resolvent estimate*, plays a crucial role. Here we have used the standard notation: $\langle x\rangle = \left(1 + |x|^2\right)^{1/2}$. They proved it using a parametrix for the time evolution. The proof was simplified and generalized by several authors with the aid of the Mourre theory ([GM], [HN], [G], [W2], etc.). Moreover, Wang proved semiclassical estimates for powers of the resolvent ([W1], [W2]):

$$\left\| \langle x\rangle^{-\alpha} (H - \lambda \pm i0)^{-n} \langle x\rangle^{-\alpha} \right\| \le C_\alpha \hbar^{-n}, \qquad \hbar > 0, \alpha > n - 1/2.$$

We also want to mention works on semiclassical resolvent estimates for high energies ([Y], [J2]).

On the other hand, motivated by works on the barrier top resonances ([BCD], [S]), the author generalized the semiclassical resolvent estimate to the simplest trapping energy, namely the barrier top energy ([N1]). In this case, the estimate has the form:

$$\left\| \langle x\rangle^{-\alpha} (H-\lambda \pm i0)^{-1} \langle x\rangle^{-\alpha} \right\| \le C_\alpha \hbar^{-2}, \qquad \hbar > 0, \alpha > 1/2,$$

where λ is the barrier top energy.

The aim of this note is to construct a semiclassical analogue of the multiple commutator method of Jensen, Mourre and Perry, and apply it to the barrier top energy and nontrapping energies. We note that for the nontrapping energy case, this was done by Wang ([W2]). Roughly speaking, our abstract result is as follows: Let A and H be a pair of self-adjoint operators satisfying certain regularity conditions (cf. (H1)–(H4) in Section 2). If, in addition, they satisfy

$$E_\Delta(H)[H, iA]E_\Delta(H) \ge c\hbar^\beta E_\Delta(H), \qquad \hbar > 0,$$

for some $1 \le \beta \le 2$, where Δ is a neighborhood of an energy E, then we can show

$$\left\| \langle A\rangle^{-\alpha} (H-E \pm i0)^{-n} \langle A\rangle^{-\alpha} \right\| \le C_\alpha \hbar^{-n\beta}, \qquad \hbar > 0, \alpha > n - 1/2.$$

$\beta = 1$ corresponds to the nontrapping case, and $\beta = 2$ to the barrier top case. We don't know any concrete examples with $1 < \beta < 2$. Even though the restriction $\beta \le 2$ doesn't seem crucial, our proof dosn't work for the case $\beta > 2$. Time-decay results in the semiclassical limit follow from the above result (Theorem 3). In particular, it follows that if $f \in C_0^\infty(\mathbf{R})$ is supported in a small neighborhood of the barrier top energy, then

$$\left\| \langle x\rangle^{-s} e^{-itH} f(H) \langle x\rangle^{-s} \right\| \le C\hbar^{-s} \langle t\rangle^{-s'}, \qquad t \in \mathbf{R},$$

for $s > s' > 0$.

This note is organized as follows: In Section 2 we state the abstract results, and it is proved in Section 4. Applications to Schrödinger operators are discussed in Section 3.

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2. Abstract Results.

Let H and A be $\hbar$-dependent self-adjoint operators on a Hilbert space $\mathcal{H}$ ($\hbar \in (0,\infty)$). We first suppose

(H1) $D(A) \cap D(H)$ is dense in $D(H)$ with respect to the graph norm.

Let $B_0 = H$. We wish to define B_j inductively by

$$B_j = [B_{j-1}, iA], \qquad j = 1, 2, \cdots,$$

at least formally. In order that we suppose

(H2) $B_1 = [H, iA]$, defined as a form on $D(H) \cap D(A)$, is extended to a bounded operator from $D(H)$ to $\mathcal{H}$. Inductively, $B_{j+1} = [B_j, iA]$, defined as a form on $D(H) \cap D(A)$, is extended to a bounded operator from $D(H)$ to $\mathcal{H}$ for any $j \geq 1$.

In this sense, H is C^∞-smooth with respect to A. We suppose the following $\hbar$-dependence of these commutators:

(H3) For each $j \geq 1$ there is $C_j > 0$ such that

$$\left\| B_j (H+i)^{-1} \right\| \leq C_j \hbar^j, \qquad \hbar > 0.$$

(H4) There is $C > 0$ such that

$$\left\| (H+i)^{-1} [H, [H, iA]] (H+i)^{-1} \right\| \leq C\hbar^2, \qquad \hbar > 0.$$

In applications, (H1)–(H4) follow easily from the symbol calculus. See Section 3.

Now let us fix an energy $E_0 \in \mathbf{R}$. The next inequality, a semiclassical variation of the Mourre estimate, is essential. Let $\beta \geq 1$.

(H5:β) There is an interval $\Delta \ni E_0$ and $C > 0$ such that

$$E_\Delta(H)[H, iA] E_\Delta(H) \geq C\hbar^\beta E_\Delta(H), \qquad \hbar > 0,$$

where $E_\Delta(H)$ is the spectral projection of H and Δ.

We prove the next theorem in Section 4.

THEOREM 1. *Suppose (H1)–(H5:β) with $1 \leq \beta \leq 2$. Then there is an interval $\Delta \ni E_0$ satisfying the following: Let $n \geq 1$ an integer, and let $s > n - 1/2$, then for any $\lambda \in \Delta$,*

$$\lim_{\delta \to +0} \langle A \rangle^{-s} (H - \lambda \pm i\delta)^{-n} \langle A \rangle^{-s} \equiv \langle A \rangle^{-s} (H - \lambda \pm i0)^{-n} \langle A \rangle^{-s}$$

exists and satisfies

$$\left\| \langle A \rangle^{-s} (H - \lambda \pm i0)^{-n} \langle A \rangle^{-s} \right\| \leq C\hbar^{-n\beta}, \qquad \hbar > 0, \lambda \in \Delta. \tag{1}$$

REMARK: Condition (H4) is missing in Lemma 2.3 of [N2], but we need it even for $n = 1$ if $\beta > 1$. On the other hand, it is not necessary if $\beta = 1$ (cf. Proof of Lemma 6).

The next result on time-decay is a direct consequence of Theorem 1.

THEOREM2. *Suppose (H1)–(H5:β) with $1 \le \beta \le 2$. Then there is an interval $\Delta \ni E_0$ such that for any $f \in C_0^\infty(\Delta)$ and for any constants $s > s' > 0$, $s'' > s'(\beta - 1)$,*

$$\left\| \langle A \rangle^{-s} e^{-itH/\hbar} f(H) \langle A \rangle^{-s} \right\| \le C \hbar^{-s''} \langle t \rangle^{-s'}, \qquad \hbar > 0, t \in \mathbf{R}. \tag{2}$$

PROOF: We follow the argument of Theorem 4.2 in [JMP]. Since

$$\begin{aligned} \left(\frac{d}{d\lambda}\right)^j E'_\lambda(H) &= \frac{1}{2\pi i} \left(\frac{d}{d\lambda}\right)^j \left((H - \lambda - i0)^{-1} - (H - \lambda + i0)^{-1}\right) \\ &= \frac{j!}{2\pi i} \left((H - \lambda - i0)^{-j-1} - (H - \lambda + i0)^{-j-1}\right), \end{aligned}$$

it follows from Theorem 1 that

$$\left\| \langle A \rangle^{-s} \left(\frac{d}{d\lambda}\right)^j E'_\lambda \langle A \rangle^{-s} \right\| \le C \hbar^{-\beta(j+1)}$$

if $s > j + 1/2$. By integration by parts and the functional calculus, we have

$$\begin{aligned} t^j e^{-itH/\hbar} f(H) &= \int_{-\infty}^{\infty} \left(t^j e^{-it\lambda/\hbar}\right) f(\lambda) E'_\lambda d\lambda \\ &= \int_{-\infty}^{\infty} e^{-it\lambda/\hbar} \left(-it\hbar \frac{d}{d\lambda}\right)^j (f(\lambda) E'_\lambda)\, d\lambda. \end{aligned}$$

Thus

$$t^j \left\| \langle A \rangle^{-s} e^{-itH/\hbar} f(H) \langle A \rangle^{-s} \right\| \le C \hbar^{-\beta} \hbar^{-(\beta-1)j},$$

and hence

$$\left\| \langle A \rangle^{-s} e^{-itH/\hbar} f(H) \langle A \rangle^{-s} \right\| \le C \langle t \rangle^{-j} \hbar^{-\beta} \hbar^{-(\beta-1)j}$$

if $s > j + 1/2$. Now (2) follows by interpolation. ∎

3. Applications.

Here we apply the results of Section 2 to Schrödinger operators:

$$H = -\frac{1}{2}\hbar^2 \Delta + V(x) \qquad \text{on} \quad \mathcal{H} = L^2(\mathbf{R}^d)$$

with $d \ge 1$, $\hbar > 0$. Throughout this section we assume the potential $V(x)$ satisfies the following condition:

(P) $V \in C^\infty(\mathbf{R}^d)$ and for any multi-index α,

$$\left| \left(\frac{d}{dx}\right)^\alpha V(x) \right| \le C_\alpha \langle x \rangle^{-|\alpha|}, \qquad x \in \mathbf{R}^d.$$