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Some Developments and Applications of the Abstract Mourre Theory

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1. Introduction

In 1979 Eric Mourre introduced the concept of locally conjugate operator and invented a very efficient method of proving the limiting absorption principle (L.A.P.). His ideas opened the way to a complete solution of the N-body problem: detailed spectral properties have been obtained by Perry, Sigal and Simon and asymptotic completeness has been proved by Sigal and Soffer. The abstract side of Mourre theory has been further developped by Perry, Sigal and Simon [PSS] (they eliminated an assumption on the first commutator which was annoying in applications) and by Mourre [M] and Jensen and Perry [JP] (the L.A.P was established in better spaces).

In [ABG] efforts were made in order to avoid the use of the second commutator of the hamiltonian with the conjugate operator. Optimal, in some sense, results in this direction were obtained in [BGM2] and [BG1]. In [BGM2] the space $\mathscr{G}$ which appears below is the domain of the hamiltonian and the main theorem is easy to apply in the N-body case with short-range and long-range interactions of a very general nature. In [BG1,2] the space $\mathscr{G}$ is the form-domain of the hamiltonian (the domain is not assumed invariant under the group generated by the conjugate operator, this being compensated by a stronger condition on the first commutator) and the theory is applied to pseudo-differential operators. In both cases, the L.A.P. is established in "optimal" (in some sense) spaces, which allows one to get without any further effort very good criteria for the existence and completeness of relative, local wave operators.

The main part of this article is devoted to an exposition of several applications of a version of the locally conjugate operator method which we developed in [BG1,2]. In fact, theorems 3.1 and 3.2 below are the main results got in [BG1] and in sections 4 and 5 we show their force and also fineness. In the preliminary section 2 we introduce and discuss the most important notion we have isolated, that of operator of class $\mathscr{C}^1$ with respect to a unitary group. This is a quite general property and in section 5 we show in some simple cases that it is almost impossible to be replaced by a weaker one without loosing the strong form of the L.A.P. given

[1] Lecture delivered by A. Boutet de Monvel-Berthier

in theorem 3.1. Moreover, in section 5 we show how to deal with hamiltonians with very singular interactions (this part will be treated more thoroughly in a later publication). But section 4 contains the most important results. Although their formulation is abstract, it is trivial to apply them to many-body hamiltonians. After the Nantes conference, as A. Soffer raised the problem of the spectral analysis of hard-core N-body hamiltonians, we decided to formulate, in this paper, several consequences of theorem 3.1 such as to cover non-densely defined hamiltonians (in fact we use pseudo-resolvents in place of resolvents). The particular case of hard-core N-body hamiltonians is the subject of a in-preparation-joint-paper with A. Soffer. Finally, an appendix contains a technical estimate related to Littlewood-Paley theory which seemes to us quite powerful in various situations.

2. Unitary Groups in Friedrichs Couples

In our approach, the natural framework for the "locally conjugate operator method" is a triplet $(\mathscr{G},\mathscr{H};W)$ consisting of two Hilbert spaces $\mathscr{G},\mathscr{H}$ such that $\mathscr{G}\subset\mathscr{H}$ continuously and densely, and a strongly continuous unitary one-parameter group $W = \{W_\alpha\}_{\alpha\in\mathbb{R}}$ in $\mathscr{H}$ which leaves $\mathscr{G}$ invariant: $W_\alpha\mathscr{G}\subset\mathscr{G}$ for all $\alpha\in\mathbb{R}$ .The Hilbert spaces are always complex but not necessarily separable. In our applications, $\mathscr{G}$ will be either the domain of the hamiltonian, or its form domain, or it will be just $\mathscr{H}$ (although, in this last case, the hamiltonian could be unbounded and even non-densely defined).

A triplet $(\mathscr{G},\mathscr{H};W)$ with the preceding properties will be called a *unitary group in a Friedrichs couple* , the pair of spaces $(\mathscr{G},\mathscr{H})$ being called a *Friedrichs couple.* In this section we shall fix such a system $(\mathscr{G},\mathscr{H};W)$ and we shall study some notions related to it.

Let $\mathscr{G}^*$ be the adjoint (or antidual) space of $\mathscr{G}$; identify $\mathscr{H}^*=\mathscr{H}$ by using Riesz lemma and embed as usual $\mathscr{G}\subset\mathscr{H}\subset\mathscr{G}^*$. Then define $\mathscr{G}^s=[\mathscr{G},\mathscr{G}^*]_{(1-s)/2}$ by complex interpolation for $-1\leq s\leq 1$, so that $\mathscr{G}^1=\mathscr{G}$, $\mathscr{G}^0=\mathscr{H}$ and $\mathscr{G}^{-1}=\mathscr{G}^*$. Observe that we have canonical identifications $(\mathscr{G}^s)^*=\mathscr{G}^{-s}$. We shall denote $\mathscr{X}=B(\mathscr{G},\mathscr{G}^*)$ the Banach space of continuous linear operators from $\mathscr{G}$ to $\mathscr{G}^*$ and $\|\cdot\|_{\mathscr{X}}$ its norm; observe that $\mathscr{X}$ is equipped with an isometric involution $T\mapsto T^*$. For each $s,t\in[-1,+1]$ we have canonical embeddings $B(\mathscr{G}^s,\mathscr{G}^t)\subset\mathscr{X}$. Then the norm in $\mathscr{G}^s$, resp. in $B(\mathscr{G}^s,\mathscr{G}^t)$, will be denoted $\|\cdot\|_s$, resp. $\|\cdot\|_{s,t}$, and we abbreviate $\|\cdot\|_0 = \|\cdot\|$, $\|\cdot\|_{0,0} = \|\cdot\|$.

The following fact will be often used below:

LEMMA 2.1: *Let* $\mathbf{E},\mathbf{F}$ *be Hilbert spaces such that* $\mathbf{E}\subset\mathbf{F}$ *continuously and let* $W_\alpha(\alpha)=e^{iA\alpha}$, $\alpha\in\mathbb{R}$, *be a* C_0*-group in* $\mathbf{F}$ *which leaves* $\mathbf{E}$ *invariant*: $W_\alpha\mathbf{E}\subset\mathbf{E}$

($\forall\alpha\in\mathbb{R}$).*Denote* $W_\alpha^{\mathbf{E}}=W_\alpha|_{\mathbf{E}}$ *considered as operator in* $\mathbf{E}$. *Then* $\{W_\alpha^{\mathbf{E}}\}_{\alpha\in\mathbb{R}}$ *is a* C_0-*group in* $\mathbf{E}$ *and its infinitesimal generator is the closed, densely defined operator* $A^{\mathbf{E}}$ *in* $\mathbf{E}$ *defined as the restriction of* A *to* $D(A^{\mathbf{E}})=\{u\in D(A)\cap\mathbf{E} \mid Au\in\mathbf{E}\}$.

Proof: The lemma has been proved in [ABG] under the assumption that **E**,**F** are separable. We shall reduce ourselves to this case. The only problem is to prove the continuity of $\alpha\mapsto W_\alpha u\in\mathbf{E}$ when $u\in\mathbf{E}$. Let $\mathbf{E}_0$ (resp. $\mathbf{F}_0$) be the closed subspace of E (resp.**F**) generated by $\{W_\alpha u \mid \alpha\in\mathbb{R}\}$. Then $\mathbf{E}_0\subset\mathbf{F}_0$ continuously and densely, W leaves $\mathbf{E}_0$ and $\mathbf{F}_0$ invariant and it is strongly continuous in $\mathbf{F}_0$. Moreover, $\mathbf{F}_0$ is separable because $\alpha\mapsto W_\alpha u\in\mathbf{F}_0$ is continuous and its image is a total subset of $\mathbf{F}_0$. Since $\mathbf{F}_0^*\subset\mathbf{E}_0^*$ continuously and densely, we see that $\mathbf{E}_0^*$ is separable, hence $\mathbf{E}_0$ is separable too. Now we may apply lemmas 1.1.3 and 1.1.4 from [ABG1] to $(\mathbf{E}_0,\mathbf{F}_0;W|_{\mathbf{F}_0})$. ■

Let us apply this lemma in the case of the unitary group W in the Friedrichs couple $(\mathcal{G},\mathcal{H})$. Denote A the self-adjoint operator in $\mathcal{H}$ such that $W_\alpha=e^{iA\alpha}$. The notations $W_\alpha^{\mathcal{G}}$, $A^{\mathcal{G}}$ have the same signification as in the preceding lemma. Now let $W_\alpha^{\mathcal{G}^*}=(W_{-\alpha}^{\mathcal{G}})^*\in B(\mathcal{G}^*)$. Since for a group weak and strong continuity are equivalent, $\{W_\alpha^{\mathcal{G}^*}\}_{\alpha\in\mathbb{R}}$ will be a C_0-group in $\mathcal{G}^*$; we denote $A^{\mathcal{G}^*}$ its generator (closed, densely defined operator in $\mathcal{G}^*$ such that $W_\alpha^{\mathcal{G}^*}=\exp(i\alpha A^{\mathcal{G}^*})$).

It is easily shown that $W_\alpha^{\mathcal{G}^*}|_{\mathcal{H}}=W_\alpha$ and an application of lemma 2.1 shows that A is just the restriction of $A^{\mathcal{G}^*}$ to $\{u\in D(A^{\mathcal{G}^*})\cap\mathcal{H} \mid A^{\mathcal{G}^*}u\in\mathcal{H}\}$. Interpolating between $\mathcal{G}$ and $\mathcal{G}^*$, we see that $W^{\mathcal{G}^*}$induces a C_0-group $W^{\mathcal{G}^s}$ in each $\mathcal{G}^s$, the infinitesimal generators of these groups being the natural restrictions of $A^{\mathcal{G}^*}$. It will be obvious in later arguments that no confusion arises if we drop the index which indicates the space in which the operators are considered. We summarize these facts in:

PROPOSITION 2.2: *Let* $(\mathcal{G},\mathcal{H};W)$ *be a unitary group in a Friedrichs couple. Then , for each* $\alpha\in\mathbb{R}$, *the operator* W_α *in* $\mathcal{H}$ *is continuous when* $\mathcal{H}$ *is equipped with the topology induced by* $\mathcal{G}^*$ *and, if we denote again by* W_α *its unique extension to a continuous operator on* $\mathcal{G}^*$, *the application* $\alpha\mapsto W_\alpha\in B(\mathcal{G}^*)$ *is a* C_0-*group in* $\mathcal{G}^*$ *which leaves invariant and induces a* C_0-*group in each space* $\mathcal{G}^s$. *Let* A *be the infinitesimal generator of the group* W *in* $\mathcal{G}^*$, i.e. A *is the unique closed, densely defined operator in* $\mathcal{G}^*$ *such that* $W_\alpha=e^{iA\alpha}$; *denote* $D(A;\mathcal{G}^*)$ *its domain. Then for each* $s\in[-1,+1]$, *the restriction of* A *to*

(2.1) $\quad D(A;\mathscr{G}^s) = \{u\in\mathscr{G}^s \mid u\in D(A;\mathscr{G}^*)$ *and* $Au\in\mathscr{G}^s\}$

is a closed, densely defined operator in $\mathscr{G}^s$ *which is just the infinitesimal generator of the* C_0*-group* $W_\alpha|_{\mathscr{G}^s}$.

We shall always consider $D(A;\mathscr{G}^s)$ as a Hilbert space, the norm being the graph norm associated to A in $\mathscr{G}^s$: $\|u\|_s^A = [\ \|u\|_s^2 + \|Au\|_s^2\ ]^{1/2}$. It follows from a well-known lemma of Nelson (see theorem 1.9 in [D]) that $D(A;\mathscr{G})\subset D(A;\mathscr{G}^s)\subset\mathscr{G}^s$ continuously and *densely* for all $s\in[-1,+1]$. Moreover, the operator A with domain $D(A;\mathscr{H})$ is self-adjoint in $\mathscr{H}$.

Finally, let us remark that the equality $W_\alpha^* = W_{-\alpha}$ has to be interpreted in the following sense: if $-1\leq s\leq 1$, then the adjoint of the operator $W_\alpha|_{\mathscr{G}^s}\in B(\mathscr{G}^s)$ is equal to $W_{-\alpha}|_{\mathscr{G}^{-s}}\in B(\mathscr{G}^{-s})$, the identification $(\mathscr{G}^s)^* = \mathscr{G}^{-s}$ being assumed.

Let us consider now the group of automorphisms of the Banach space $\mathscr{X}=B(\mathscr{G},\mathscr{G}^*)$ induced by W, namely $\mathscr{W}_\alpha(T)=W_\alpha T W_\alpha^*$ for $T\in\mathscr{X}$. Observe that $\alpha\mapsto\mathscr{W}_\alpha(T)\in\mathscr{X}$ is continuous only when $\mathscr{X}$ is equipped with the strong operator topology, hence $\{\mathscr{W}_\alpha\}_{\alpha\in\mathbb{R}}$ is not a C_0-group on $\mathscr{X}$. However, one has $\mathscr{W}_\alpha = e^{i\mathscr{A}\alpha}$, with $\mathscr{A}(T)=[A,T]$, in a sense which we shall explain below.

DEFINITION 2.3: *Let* $0<\theta\leq 1$. *We shall say that an operator* $T\in B(\mathscr{G},\mathscr{G}^*)$ *is of class* $C^\theta(A;\mathscr{G},\mathscr{G}^*)$, *and we shall write* $T\in C^\theta(A;\mathscr{G},\mathscr{G}^*)$, *if the function* $\alpha\mapsto\mathscr{W}_\alpha(T)\in\mathscr{X}$ *is Hölder continuous of order* θ, *i.e. there is* $c<\infty$ *such that* $\|W_\varepsilon T W_\varepsilon^* - T\|_{\mathscr{X}} \leq c|\varepsilon|^\theta$ *for* $|\varepsilon|\leq 1$. *For* $\theta=+0$ *we replace Hölder-continuity by Dini-continuity, more precisely we write* $T\in C^{+0}(A;\mathscr{G},\mathscr{G}^*)$ *if* $\int_0^1 \|W_\varepsilon T W_\varepsilon^* - T\|_{\mathscr{X}}\ \varepsilon^{-1}d\varepsilon<\infty$.

Remark that we could replace here $W_\varepsilon T W_\varepsilon^* - T$ by the commutator $[T,W_\varepsilon]=TW_\varepsilon - W_\varepsilon T=(W_\varepsilon T W_\varepsilon^* - T)W_\varepsilon$. One can refine the notion and define $T\in C^\theta(A;\mathscr{G}^s,\mathscr{G}^t)$ for some $-1\leq s,t\leq 1$ by replacing the norm $\|\cdot\|_{\mathscr{X}}$ with the norm $\|\cdot\|_{s,t}$.

If $T:\mathscr{G}\to\mathscr{G}^*$ is a linear continuous operator, we shall denote $[A,T]\equiv-[T,A]$ the continuous sesquilinear form on $D(A;\mathscr{G})$ defined by the formula $\langle u|[A,T]v\rangle = \langle Au|Tv\rangle - \langle u|TAv\rangle$. Taking into account that W is a C_0-group in $\mathscr{G}$