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NEAR—COHOMOLOGY OF HILBERT COMPLEXES AND TOPOLOGY OF NON—SIMPLY CONNECTED MANIFOLDS.

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Introduction

In an earlier paper [5] we introduced some new homotopy invariants of compact non–simply connected manifolds (possibly with boundary) or finite CW–compexes. In terms of these invariants the heat kernel invariants of closed non–simply connected manifolds [9] (see also [4]) can be expressed and thus their homotopy invariance can be proved.

Note that both invariants in [5] and [9] are expressed in terms of L^2–de Rham complex on the universal covering, using the deck transformation action of the fundamental group in differential forms. The use of the combinatorial Laplacians leads to the same invariants as was proved by A. Efremov [3].

In this paper we follow the abstract setting from [5] and give a refined formulation of the abstract result there. This leads to a new notion of near–cohomology for Hilbert complexes. We take a special family of quadric cones depending on a small positive parameter and consisting of cochains which have coboundaries which are small with respect to the distance of the cochains to the space of all cocycles. Heuristically this means that we take cochains with small coboundaries modulo cochains close to cocycles. (Instead of cochains close to cocycles we could also take cochains close to coboundaries which would remind cohomology more but it just adds cohomology as a direct summand.) Near–cohomology are germs of such families of quadric cones modulo an equivalence relation which naturally arises if we consider homotopy equivalence of Hilbert complexes with morphisms given by bounded linear operators. Then near–cohomology becomes a homotopy invariant.

Adding a von Neumann algebra structure to the Hilbert complex we can transform near–cohomology to a set of positive–valued functions of the small parameter up to an equivalence. These functions are defined as maximal von Neumann dimensions of linear spaces which belong to the cones. The equivalence is given by estimates of these functions with dilatated arguments.

Applying these constructions to the de Rham L^2–complex on the universal covering of a compact manifolds (with the von Neumann algebras consisting of operators commuting with deck transformations on differential forms) we obtain invariants which were introduced and studied in [5].

Note that the idea that there may be topology invariants lying near cohomology was first formulated in [8].

1. Hilbert complexex and their near–cohomology.

A. Let us consider a sequence

$$E:\ 0 \longrightarrow E_0 \xrightarrow{d_0} E_1 \longrightarrow \ldots \longrightarrow E_k \xrightarrow{d_k} E_{k+1} \longrightarrow \ldots \xrightarrow{d_{N-1}} E_N \longrightarrow 0,$$

where E_k is a Hilbert space and the differential $d_k:\ E_k \longrightarrow E_{k+1}$ is a closed densely defined linear operator (with the domain $D(d_k)$). This sequence is called a *Hilbert complex* if $d_{k+1} \circ d_k = 0$ on $D(d_k)$ or, equivalently, Im $d_k \subset$ Kerd_{k+1}. Note that Ker d_k is always a closed linear subspace in E_k.

Let E' be another Hilbert complex of the same length N (if the lengths differ then we can always formally extend the shorter complex by adding zero spaces in the end; so for the sake of simplicity we shall always suppose that all complexes have the same length N). The corresponding spaces and differentials will be denoted E'_k and d'_k.

Definition 1.1. A *morphism* $f:\ E \longrightarrow E'$ of the Hilbert complexes is a collection of *bounded* linear operators $f_k:\ E_k \longrightarrow E'_k$ such that

$$f_{k+1}d_k \subset d'_k f_k,$$

which means that $f_{k+1}d_k = d'_k f_k$ on $D(d_k)$. In particular we require that $f_k(D(d_k)) \subset D(d'_k)$.

If $f : E \longrightarrow E'$ and $g : E' \longrightarrow E''$ are two morphisms of Hilbert complexes then their composition $g \circ f : E \longrightarrow E''$ is a morphism defined as the collection of compositions $g_k \circ f_k$, $k = 0, 1, \dots, N$.

Definition 1.2. Let $f, g : E \longrightarrow E'$ be two morphisms of the same Hilbert complexes. A *homotopy* (between f and g) is a collection T of *bounded* linear operators $T_k : E_k \longrightarrow E'_{k-1}$ such that

$$f_k - g_k - T_{k+1}d_k \subset d'_{k-1}T_k, \ k = 0, 1, \dots, N,$$

or equivalently, $f_k - g_k = T_{k+1}d_k + d'_{k-1}T_k$ on $D(d_k)$ (in particular this means that $T_k(D(d_k)) \subset D(d'_{k-1})$). If there exists a homotopy between morphisms f and g then f and g are called *homotopic* and we denote it as $f \sim g$. (It is easy to check that being homotopic is really an equivalence relation between morphisms.)

Hilbert complexes E, E' are called *homotopy equivalent* if there exists morphisms $f : E \longrightarrow E'$ and $g : E' \longrightarrow E$ such that $g \circ f \sim \mathrm{Id}_E$, $f \circ g \sim \mathrm{Id}_{E'}$ where Id_E and $\mathrm{Id}_{E'}$ are identity morphisms of the corresponding Hilbert complexes. We shall denote the homotopy equivalence between E and E' as $E \sim E'$.

Definition 1.3. E is called a *retract* of E' if there exist morphisms $f : E \longrightarrow E'$ and $g : E' \longrightarrow E$ such that $g \circ f \sim \mathrm{Id}_E$. In this case f (resp. g) is called a *homotopy inclusion* (resp. *homotopy retraction*) map.

Remark. Cohomology spaces $H^k(E) = \mathrm{Ker}\ d_k / \mathrm{Im}\ d_{k-1}$ and reduced cohomology spaces $\overline{H}^k(E) = \mathrm{Ker}\ d_k / \overline{\mathrm{Im}\ d_{k-1}}$ are homotopy functors in the category of Hilbert complexes with morphisms and homotopy as before.

B. Now let us introduce the following quadric cones, depending on the degree k and on a positive parameter λ:

$$B_\lambda^{(k)} = \{\omega | \omega \in E_k/\text{Ker } d_k, \| d_k\omega \| \le \lambda \| \omega \|_{\text{mod Ker } d}\},$$

where $\| \omega \|_{\text{mod Ker } d}$ is the norm in the quotient space $E_k/\text{Ker } d_k$, $\| d_k\omega \|$ means the norm of $d_k\omega$ in E_{k+1}. It is understood that in this definition we should only take co–sets in $E_k/\text{Ker } d_k$ defined by elements $\omega \in D(d_k)$ to make $d_k\omega$ well defined. So $B_\lambda^{(k)}$ becomes a conic set in the Hilbert space $E_k/\text{Ker } d_k$ which can also be identified with $(\text{Ker } d_k)^\perp$ (the orthogonal complement of Ker d_k in E_k).

Lemma 1.4. Let $\mathcal{H}_1, \mathcal{H}_2$ be Hilbert spaces, $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a closed linear operator with the domain $D(A)$. Then for every $\lambda > 0$ the set

$$C_{\lambda,A} = \{x | x \in D(A), \ \| Ax \| \le \lambda \| x \|\}$$

is closed in $\mathcal{H}_1$.

Proof. Suppose that x is in the closure of $C_{\lambda,A}$. Without loss of generality we may assume that $\| x \| = 1$. Then we easily obtain that there exist $x_\gamma \in C_{\lambda,A}$ such that

$$\lim_\Gamma \| x_\gamma - x \| = 0, \ \| Ax_\gamma \| \le \lambda \| x_\gamma \|, \ \gamma \in \Gamma,$$

where Γ is a directed set. Taking a cofinal subset of Γ we may further suppose that $\| x_\gamma \| \le 1+\varepsilon$ whatever fixed $\varepsilon > 0$. Changing Γ again we may suppose that there exists $w - \lim_\Gamma Ax_\gamma = y$ (weak limit is taken in $\mathcal{H}_2$). Then we have

$$\| y \| \le \liminf_\Gamma \| Ax_\gamma \| \le \lambda \liminf_\Gamma \| x_\gamma \| \le \lambda \| x \|$$

Now the pair $\{x, y\}$ is in the weak closure of the graph of the operator A in $\mathcal{H}_1 \times \mathcal{H}_2$. The graph is a closed linear subspace, hence it is weakly closed. Therefore $x \in D(A)$ and $y = Ax$. Hence $x \in C_{\lambda,A}$ as required. □

Applying Lemma 1.4 to $\mathcal{H}_1 = E_k/\text{Ker } d_k$, $\mathcal{H}_2 = E_{k+1}$ and $A = d_k$ we see that $B_\lambda^{(k)}$ is a closed cone in $E_k/\text{Ker } d_k$ for every