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0. Introduction

In **[HS]** we introduced a class of semi-classical Schrödinger operators of the form $-\frac{1}{2}h^2\Delta + V^{(m)}$ on $\mathbb{R}^m$ for $m = 1, 2, \dots$, where $V^{(m)}$ satisfy various assumptions, implying in particular convexity. If $\mu(m;h)$ denotes the first eigenvalue, we showed among other things that $\mu(m;h)/m$ tends to a limit $\mu(\infty;h)$ when $m \to \infty$ and that :

$$\mu(m;h)/m - \mu(\infty;h) = \mathcal{O}(h/m) \ . \tag{0.1}$$

We also proved (by adapting the methods of **[S1, 2]**) that $\mu(\infty;h)$ has an asymptotic expansion $\sim h(\mu_0 + \mu_1 h + \dots)$, when $h \to 0$. One element of the proof was the use of certain WKB-expansions, more precisely, we showed that if $h(\mu_0(m) + \mu_1(m)h + \dots)$ is the formal asymptotic expansion of $\mu(m;h)$, then $\mu_k(m)/m \to \mu_k$ when $m \to \infty$ with an exponential rate of convergence. A natural question is then wether (0.1) can be improved to :

$$\mu(m;h)/m - \mu(\infty;h) = \mathcal{O}(e^{-\kappa m}) \tag{0.2}$$

for some suitable $\kappa > 0$.

In this work, we establish estimates of the form (0.2) for certain sequences of $V^{(m)}$. A general result of this type is given in Theorem 3.1, and in Theorem 4.1 we obtain a better rate of exponential convergence for a somewhat more restricted class of potentials. In particular, we study in section 5 the same sequence of potentials related to statistical mechanics as in **[HS]**, and show that we get exponential convergence with a rate which seems to be optimal.

In **[HS]** we obtained exponential convergence at the level of WKB-eigenvalues by introducing exponential weights in the study of certain Hessians of the logarithm of certain WKB approximations to the first eigenfunction. These estimates were obtained by adapting the WKB-constructions in the complex domain of [**S1, 2**], and by introducing certain exponential weights in these estimates. In the present work, we also establish exponentially weighted estimates of certain Hessians of the logarithm of the first eigenfunction, but this time we work with the exact first eigenfunctions, and inspired by the appendix b in **[SiWYY]**, we use systematically the maximum principle in order to obtain these estimates. In particular, we never use any small h expansions, and our results are uniform in h.

The plan of the paper is the following : In section 1, we make some estimates for the log. of the first eigenfunction near $|x| = \infty$, in the case when the potential is a compactly supported perturbation of $\frac{1}{2}x^2$. These estimates, which are not necessarily uniform with respect to the dimension, form a preparation for the more refined estimates that we obtain in section 2. In section 3 we get a first result about the validity of (0.2).

In section 4, we start by examining a sequence of simple quadratic potentials, and we see that Theorem 3.1 does not give the optimal κ in this case. Then after some further exponential estimates in the style of section 2, we obtain the sharper Theorem 4.1, which is valid under somewhat different assumptions. In section 5, we apply this result to the model problem from statistical mechanics already studied in **[HS]**, and establish (0.2) with a set of κ which seems to be optimal.

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1. Some estimates for the exterior Dirichlet problem for the harmonic oscillator

Let B be an open a ball in $\mathbb{R}^n$ centered at 0. Then the Dirichlet realization P of $-\Delta+x^2$ in $\mathbb{R}^n\setminus B$ has discrete spectrum. Choose $\mu\in\mathbb{R}$ such that $x^2-\mu>0$ in $\mathbb{R}^n\setminus B$. Then μ is also below the infimum of the spectrum of the operator P just defined, and we let $K : C^\infty(\partial B)\to C^\infty(\mathbb{R}^n\setminus B)$ be the operator such that $u=Kv$ belongs to the domain of P outside a compact set and solves the problem:

$$(1.1)\qquad (-\Delta+x^2-\mu)u=0\ ,\quad u|_{\partial B}=v\ .$$

Using weighted L^2 estimates we see that $\partial^\alpha Kv(x)\to 0$, $|x|\to\infty$, for every α. Using the maximum principle we then have that $v\geq 0\Rightarrow Kv\geq 0$. This implies that if $v_1\leq v_2$ then $Kv_1\leq Kv_2$, and also $Kv\leq\sup v$, if $\sup v\geq 0$, $Kv\geq\inf v$ if $\inf v\leq 0$. Of particular interest is $K(1)$ which is a radial function $u_0=u_0(|x|)$, with:

$$(1.2)\qquad (-\partial_r^2-((n-1)/r)\partial_r+r^2-\mu)u_0(r)=0\ ,\ u_0(1)=1\ .$$

Here and in the following we assume (without loss of generality) that B is the unit ball. Writing $u_0=r^{-(n-1)/2}f(r)$, we know that f is in $L^2([1,\infty[,dr)$ and satisfies the Schrödinger equation:

$$(1.3)\qquad (-\partial_r^2+r^2+(n-1)(n-3)/4r^2-\mu)f=0\ ,\quad f(1)=1\ .$$

We can construct $\varphi(r)$ with

$$(1.4)\qquad \varphi'(r)\sim r+a_{-1}r^{-1}+a_{-3}r^{-3}+\ldots,\ r\to+\infty\ ,$$

such that

$$(1.5)\qquad (-\partial_r^2+r^2+(n-1)(n-3)/4r^2-\mu)(e^{-\varphi(r)})=e^{-\varphi(r)}\widetilde{R}(r)\ ,$$

where $\widetilde{R}$ is rapidly decreasing with all its derivatives when $r\to+\infty$. Actually we solve asymptotically the equation $(\varphi')^2-\varphi''=r^2+(n-1)(n-3)/4r^2-\mu$, and it is a routine procedure to verify that $f=e^{-(\varphi+R)}$, with $\partial^\alpha R=\mathcal{O}(r^{-\infty})$ for every $\alpha>0$. Replacing φ by $\varphi+R$, we still have (1.4). With $g(r)=\varphi(r)+((n-1)/2)\log r$, we get:

$$(1.6)\qquad u_0=e^{-g(|x|)}\ .$$

Here we note that $\partial_{x_\nu}g(|x|)=g'(|x|)x_\nu/|x|=x_\nu+\mathcal{O}(1/|x|)$,

$$(1.7)\qquad \partial_{x_\mu}\partial_{x_\nu}g=\delta_{\nu,\mu}+\mathcal{O}(|x|^{-2})\ .$$

Let now $v \in C^\infty(\mathbb{S}^{n-1})$ be strictly positive everywhere and let $u = Kv$. If $0 < v_{\min} < v_{\max}$ denote the infimum and the supremum of v, then we have:

$$v_{\min} u_0 \leq u \leq v_{\max} u_0 \ , \tag{1.8}$$

and hence:

$$u = e^{-g(|x|)+k(x)} \ , \tag{1.9}$$

where k is a bounded function. If the vectorfield ν is an infinitesimal generator of a rotation of $\mathbb{S}^{n-1}$, and if we extend the definition of ν to $\mathbb{R}^n$ by means of polar coordinates, (r,θ), $x = r\theta$, then $\nu \circ K = K \circ \nu$. Since v is C^∞, it follows that $\partial_\theta^\alpha u = \mathcal{O}(1) e^{-\psi(r)}$ for every α. We conclude that

$$\partial_\theta^\alpha k = \mathcal{O}(1) \text{ for every } \alpha \ . \tag{1.10}$$

We also need to control some radial derivatives of k. Writing

$$\Big(- \partial_r^2 - ((n-1)/r)\partial_r + r^2 - \mu - r^{-2}\Delta_\theta\Big)(u_0(r)e^k) = 0 \ ,$$

and using (1.2), we get:

$$\Big(\partial_r^2 + (2(\partial_r u_0)/u_0 + (n-1)/r)\partial_r\Big)(e^k) = -r^{-2}\Delta_\theta e^k \ . \tag{1.11}$$

Here $\partial_\theta^\alpha(r^{-2}\Delta_\theta(e^k)) = \mathcal{O}(r^{-2})$, and we have $2(\partial_r u_0)/u_0 = -2\partial_r g$, so (1.11), (1.5) imply that

$$(\partial_r - f(r))\partial_r(e^k) = -r^{-2}\Delta_\theta(e^k) = \mathcal{O}(r^{-2}) \ , \tag{1.12}$$

where $f(r) = 2r + \mathcal{O}(1/r)$, $f'(r) = 2 + \mathcal{O}(1/r^2)$ etc. Let $F(r) = \displaystyle\int_1^r f(t)\,dt$. Then

$$\partial_r e^k = -\int_r^{+\infty} e^{F(r)-F(s)}\mathcal{O}(s^{-2})\,ds + Ce^{F(r)} \ . \tag{1.13}$$

The first term is $\mathcal{O}(r^{-3})$ since $F(r) - F(s) \sim r^2 - s^2 \leq 2r(r-s)$, for $s \geq r$, and since we know that $\partial_r e^k$ cannot tend to $+\infty$ or $-\infty$, when $r \to \infty$, we conclude that $C = 0$ in (1.13), and hence:

$$\partial_r e^k = \mathcal{O}(r^{-3}) \ . \tag{1.14}$$

More generally,

$$\partial_r \partial_\theta^\alpha e^k = \mathcal{O}(r^{-3}) \ . \tag{1.15}$$