Astérisque

B. R. VAINBERG

Scattering of waves in a medium depending periodically on time

Astérisque, tome 210 (1992), p. 327-340
<http://www.numdam.org/item?id=AST_1992__210__327_0>

SCATTERING OF WAVES IN A MEDIUM DEPENDING PERIODICALLY ON TIME

B. R. VAINBERG

I. INTRODUCTION

We obtain the asymptotic behaviour as $t \to \infty, |x| \leq a < \infty$ of solutions of exterior mixed problems for hyperbolic equations and systems when the boundary of a domain and coefficients of the equations depend periodically on time. Our method can be regarded as an alternative one to the Lax-Phillips scattering theory. Using the Lax-Phillips method we have to construct at first waves operators and a scattering matrix. Then we study some analytic properties of the scattering matrix and some properties of a special Lax-Phillips semigroup $Z(t)$ and then we derive asymptotic behavior of solutions of the exterior mixed problem as $t \to \infty$. In our direct method at first we find the asymptotic behavior of the solution of the exterior mixed problem. Unlike Lax-Phillips we do it without using any abstract result on spectral representation, outgoing and ingoing subspaces and so on. Then we obtain existence of the wave operators and the scattering operator. In fact, it is not a difficult problem if you know asymptotic behavior of the solutions.

Both of these methods were constructed earlier in the stationary case, when the domain and coefficients of the equations did not depend on time (there are references in [6]). Recently a few papers by J. Cooper and W. Strauss appeared which contain some results of Lax-Phillips theory for scattering of waves by a body moving periodically in t ([1],[2],[3]). Another method of research of this problem is based on the theorem of RAGE type and is suggested by V. Petkov [4]. These authors proved the existence of a scattering operator for wave equation in exterior of a body which depends periodically on t if $n \geq 3$ and obtained asymptotic behavior of solutions of this problem for odd n. They also studied hyperbolic systems of first order when dimension n is odd. Our method gives the possibility to study general time periodic systems of any order and moreover the dimension of the space can be arbitrary and the energy of solution can be unbounded with respect to time. Some of the proofs given below are very concise. The omitted details can be reconstructed with the help of [7], [8], [9].

II. ASYMPTOTIC BEHAVIOR OF SOLUTIONS

Let $x \in I\!R^n$, $\partial_t = \partial/\partial_t$, $\partial_x = (\partial/\partial x_1, \cdots, \partial/\partial x_n)$, $\Omega \in I\!R^{n+1}_{(t,x)}$ be the exterior of the cylinder with a curvelinear boundary which depends periodically on t. Let $u = (u^{(1)}, ..., u^{(\ell)})$,
$L = L(t, x, \partial_t, \partial_x) = \{L_{i,j}\}$ be a hyperbolic $\ell \times \ell$ matrix. We consider the exterior mixed problem

$$(1) \qquad \begin{cases} Lu = 0, \quad (t,x) \in \Omega, \quad t > \tau; \qquad Bu|_{\partial\Omega} = 0, \quad t > \tau; \\ \partial_t^j u|_{t=\tau} = f_j, \quad 0 \le j \le m-1, \quad x \in \Omega_\tau = \Omega \bigcap \{t = \tau\}. \end{cases}$$

Here $B = B(t, x, \partial_x)$ is a boundary operator of general type, $m = \max\limits_{i,j}$ ord $L_{i,j}$.

The main problem of this part of the article is the following. Let $f = (f_0, ..., f_{m-1})$ be a function with a compact support. The asymptotic behavior of solution u is to be found when $t \to \infty$ and x is bounded, that is the initial data are localized in space and the solution at large t is of interest only in the limited part of the space.

We fix an arbitrary constant a for which $\partial\Omega \subset \{(t,x) : |x| < a - 1\}$, condition H_1 is satisfied and $f = 0$ when $|x| > a$.

Conditions.

H_1. The medium is homogeneous in the neighborhood of infinity, that is $L = L_0(\partial_t, \partial_x)$ when $|x| > a$, where L_0 is a homogeneous matrix with constant coefficients.

H_2. The problem (1) is time periodic, that is $\Omega_{t+T} = \Omega_t$ and coefficients of the operators L and B are periodic functions with respect to t with the same period T.

H_3. The problem (1) is correct and Duhamel principle is valid.

Let $C_a^\infty(\bar\Omega_\tau), C_a^\infty(\bar\Omega)$ be spaces of infinitely smooth functions in $\bar\Omega_\tau$ or $\bar\Omega$ which are equal to zero when $|x| > a$; $H^s(D)$ be a Sobolev space of functions in domain D, $H^s_{loc}(\bar D)$ be a space of functions in the domain D belonging to $H^s(V)$ for any bounded domain $V \subset D$;

$\psi \in H^{s,A}$ if $\exp(At)\psi \in H^s(\Omega)$;
$\psi \in H^{s,A}_{a,0}$ if $\psi \in H^{s,A}$ and $\psi = 0$ when $|x| \geq a$ *or* $t < 0$.

If $\nu = (\nu_0, ..., \nu_{m-1})$, then we denote $H^\nu(\Omega_\tau) = \sum_{0 \leq j \leq m-1} H^{\nu_j}(\Omega_\tau)$. Let $f = (f_0, \ldots, f_{m-1}) \in F_\tau$ if $f_j \in C^\infty_a(\bar{\Omega}_\tau)$ and compatibility conditions are satisfied, that is there exists $w \in H^m_{loc}(\bar{\Omega})$ for which boundary and initial data of problem (1) are valid.

We shall use the same notation for the space of functions and vector-functions if the latter is a direct product of n copies of the space of functions. At last let H(ν) be the closure of the space F_τ with respect to the norm of the space $H^\nu(\Omega_\tau)$.

The correctness of the problem (1) means that it has the unique solution $u \in H^m_{loc}(\bar{\Omega} \bigcap \{t \geq \tau\})$ for any $f \in F_\tau$ and there are $\nu_j, q \in I\!R$ such that the operator

$$U_\tau : f \to \begin{cases} u, & t > \tau \\ 0, & t < \tau, \end{cases} \quad f \in F_\tau$$

has the following continuous extension: $U_\tau : H(\nu) \to H^q_{loc}(\bar{\Omega})$.

According to Duhamel principle there exist $A_0(s)$ such that the problem

$$Lw = g, \quad (t,x) \in \Omega; \quad Bw|_{\partial\Omega} = 0; \quad w = 0 \quad \text{when} \quad t < 0$$

is uniquely solvable in the space $H^{s,A}$ for any $g \in H^{s,A}_{a,0}$ if $s \geq m, A \geq A_0(s)$. Besides the operator

$$V : H^{s,A}_{a,0} \to H^{s,A}, \quad Vg = w, \quad s \geq m, \quad A \geq A_0(s) \tag{2}$$

is bounded and

$$w(t,x) = \int_0^t u(t,\tau,x)\, d\tau\,.$$

Here u is the solution of the problem (1) with $f = Pg(\tau,\cdot)$, where $Pg = (0, ..., 0, g(\tau,x))$. It is implied that $Pg \in H(\nu)$ if $g \in H^{m,A}_{a,0}$.

The condition H_3 means that the boundary of the body must not move too quickly. For example, for the wave equation the velocity of the moving boundary must be lower than the velocity of propagation of waves in the medium. In this case the condition H_3 is satisfied for all the basic problems for wave equation.

In the case of general hyperbolic equations and systems we change the variables $(t,x) \to (t,y)$, $y = y(t,x)$ so that Ω could take the form of the straight cylinder. The velocity of the moving boundary must be such that the system in the new variables remains hyperbolic at t. Then the condition H_3 is satisfied if boundary operators satisfy uniform Shapiro-Lopatinsky condition.

H_4. Non-trapping condition. It means the following.

Let $E = E(t, \tau, x, x^0)$ be the Schwartz kernel of the operator U_τ, that is E is Green matrix of the problem (1). It is supposed that there exists such a function $T(\rho)$, that E is infinitely smooth when $|x|, |x^0| < \rho, \quad t - \tau > T(\rho)$. This condition is equivalent to the following: all the bicharacteristics are outgoing to infinity when t tends to infinity.

H_5. The operator L_0 has no waves with zero propagation velocity, that is $detL_0(0, \sigma) \neq 0$ when $\sigma \neq 0$. One can give up this condition in the same way as it was done in the stationary case in |5|.

Let (1^0) denote the problem (1), when $\tau = 0$.

THEOREM 1. *Let the conditions $H_1 - H_5$ be satisfied, $f \in H(\nu)$. Then there exists a sequence of complex points k_j which are called the scattering frequencies and integers p, q, p_j and periodic on t functions $u_0(t, x), u_{j,l}(t, x) \in C^\infty$ with period T such that*

1) $-\pi/\mathrm{T} \leq \mathrm{Re} k_j < \pi/\mathrm{T}, \quad \mathrm{Im} k_{j+1} \leq \mathrm{Im}\, k_j, \quad \mathrm{Im}\, k_j \to -\infty \quad as \quad j \to \infty$

2) *If n is odd then the solution of the problem (1^0) has the following expansion*

$$u = \sum_{j=1}^{N} \sum_{t=0}^{p_j} C_{j,l} u_{j,l}(t, x) t^l \exp{(-ik_j t)} + u_N, \tag{3}$$

where there exist λ and $C = C(a, N, j, \alpha)$ such that

$$|\partial_t^j \partial_x^\alpha u_N| \leq C t^\lambda \exp{(\mathrm{Im} k_{N+1} t)} \|f\|_{H(\nu)}, \quad |x| \leq a, \quad t \to \infty. \tag{4}$$

3) If *n is even then*

$$u = \sum_{\mathrm{Im}\, k_j \geq 0} \sum_{l=0}^{p_j} C_{j,l} u_{j,l}(t, x) t^l \exp(-ik_j t) + C_0 u_0(t, x) t^p ln^q t + w, \tag{5}$$

where $C_{j,l} = C_{j,l}(f), C_0 = C_0(f)$ and

$$|\partial_t^j \partial_x^\alpha w| \leq C |\partial_t^j (t^p ln^{q-1} t)| \; \|f\|_{H(\nu)}, \quad |x| \leq a, \quad t \to \infty. \tag{6}$$

<u>Remark.</u> The scattering frequencies k_j belonging to the upper half plane correspond to the exponentially growing terms. They are finite in number. The scattering frequencies k_j belonging to the real axis correspond to the terms,