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Long Range Scattering and the Stark Effect

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1 Introduction.

In this Article we discuss long range quantum mechanical scattering in the presence of a constant electric field. The electric field is assumed to be of unit strength in the $\mathbf{e}_1 = (1, 0, \ldots, 0)$ direction of n-dimensional space, $\mathbf{R}^n$. The corresponding Hamiltonian for a quantum particle of unit mass is $H_0 = -(1/2)\Delta - x_1$, with $\Delta = \sum_{j=1}^n \partial^2/\partial x_j^2$. ($H_0$ is essentially self adjoint (as an operator on $L^2(\mathbf{R}^n)$) on the Schwartz space of rapidly decreasing smooth functions.) A second Hamiltonian $H = H_0 + V$ is regarded as a perturbation of H_0 by a potential V. The potential $V = V_S + V_L$ consists of a "short range" term V_S and a "long range" term V_L. More precisely,

SR Hypothesis. *V_S is a symmetric operator, $V_S(H_0 + i)^{-1}$ is a compact operator and*

$$\int_1^\infty \|F(x_1 > r^2)V_S(H_0 + i)^{-1}\| \, dr \; < \; \infty$$

where $F(\cdot)$ is multiplication by the characteristic function of the indicated set.

LR Hypothesis. *$V_L(x)$ is real valued on $\mathbf{R}^n$, infinitely differentiable and for some $\epsilon > 0$ and for every multi-index α*

$$\begin{aligned} |D^\alpha V_L(x)| \; &< \; C_\alpha \langle x_1 \rangle^{-|\alpha|/2-\epsilon} \\ |D^\alpha V_L(x)| \; &< \; o(1) \quad \text{as } |x| \to \infty. \end{aligned}$$

Here $\langle x_1 \rangle^2 = 1 + x_1^2$ and $D = -i\nabla$.

Example. If V_S is multiplication by a real valued function

$$V_S(x) = \{\chi(x_1)(1 + x_1^2)^{-\sigma/2} + \chi(-x_1)(1 + x_1^2)^{1/2}\}\tilde{V}_S(x)$$

where $\sigma > 1/2$ and $\tilde{V}_S = o(1)$ as $|x| \to \infty$ and $\tilde{V}_S$ is bounded and measurable and where

$$\chi(x_1) = \begin{cases} 1 & \text{if } x_1 > 1 \\ 0 & \text{if } x_1 < -1 \end{cases} \tag{1.1}$$

then V_S verifies the above short range hypothesis. (See Yajima [16]; local singularities may also be allowed.) The long range assumption is satisfied if, for some $0 \leq \alpha, \beta \leq 1/2$ and some real b_1 and b_2,

$$V_L(x) \;=\; \langle x\rangle^{-\epsilon}\cos(b_1|x_1|^\alpha)\cos(b_2|x|^\beta).$$

In general these assumptions assure that $V(H_0+i)^{-1}$ is compact so that H is self adjoint on the domain of H_0. (Perry's book [14] is a good general reference.)

Introduce now the wave operators. Dollard's [3] *modified* wave operators W_D^- and W_D^+ are defined by

$$W_D^{\pm} = \text{s-}\lim_{t\to\pm\infty} e^{itH}e^{-itH_0}e^{-iX_D(t)} \tag{1.2}$$

where "s-lim" indicates that the limit is taken in the strong operator topology. The "modifier" $e^{-iX_D(t)}$ was first introduced by J.D. Dollard [3] in the case of no electric field ($H_0 = -\Delta/2$) to extend the usual scattering theory which was based on the *Møller* wave operators,

$$W^{\pm} = \text{s-}\lim_{t\to\mp\infty} e^{itH}e^{-itH_0}, \tag{1.3}$$

to the case $V = V_L$ was the Coulomb potential ($V_L(x) = C/|x|$, for C a constant). An alternative choice of wave operators, are the *two Hilbert space* wave operators

$$W^{\pm}(J^{\pm}) = \text{s-}\lim_{t\to\mp\infty} e^{itH}J^{\pm}e^{-itH_0} \tag{1.4}$$

where $J^{\pm}$ are bounded operators conveniently chosen (as in §2 below.) The application of these operators to study long range scattering is due to Isozaki-Kitada [8] (who called $J^{\pm}$ "time independent modifiers") and Kitada-Yajima [12] who considered the case of no electric field. The two Hilbert space wave operators have certain technical advantages over the modified wave operators but the latter are the historical vehicle for studying long range scattering and are important for proving the non-existence of $W^{\pm}$; see Theorem 3.1 below. Each of the wave operators (for example W_D^+) is said to be (strongly asymptotically) *complete* if its range is the *subspace* $L^2(\mathbf{R}^n)_c$ *of continuity* of H. ($L^2(\mathbf{R}^n)_c$ is the orthogonal complement of all the eigenvectors of H.) Each wave operator (W_D^+, to be specific) is said to *intertwine* H and H_0 if

$$e^{-itH}W_D^+ = W_D^+e^{-itH_0}.$$

To state our results we must introduce the "modifiers." For the two Hilbert space wave operators we choose [8]

$$J^{\pm}u(x) = \int e^{ix\cdot\xi + i\theta^{\pm}(x,\xi)}\hat{u}(\xi)d\xi \tag{1.5}$$

where $\hat{u}$ denotes the Fourier transform of u and $\theta^{\pm}$ are smooth real valued functions to be specified in §2 below. ($J^{\pm}$ are not unique.) Here and below integrals are understood to be over all of $\mathbf{R}^n$ unless otherwise indicated and $đ\xi = (2\pi)^{-n/2}\, d\xi$.

Theorem 1.1 *Hypotheses LR and SR imply that the two Hilbert space wave operators $W^{\pm}(J^{\pm})$ exist and are complete and are isometries that intertwine H and H_0. Moreover H has no singularly continuous spectrum and its eigenvalues are discrete and of finite multiplicity.*

Dollard's time dependent modifier can be defined as follows: Let $X_D(t)$ be Fourier equivalent to multiplication by a real valued function, X where

$$X(\xi_2, \dots, \xi_n, t) = \int_0^{\pm t} V_L(\tau Y(\xi_2, \dots, \xi_n, \tau) + (\tau^2/2)\mathbf{e}_1)\, d\tau \qquad (1.6)$$

for $\pm t > 0$ and where Y is some smooth function of $n-1$ momentum variables plus time (t) taking values in $\mathbf{R}^n$ such that the first component $Y_1(\xi_2, \dots, \xi_n, t) \equiv 0$ and

$$\begin{aligned} |D^{\beta}_{\xi_\perp}(Y(\xi_2, \dots, \xi_n, t) - \xi_\perp)| &= O(|t|^{-\epsilon}); \\ |\frac{d}{dt} Y(\xi_2, \dots, \xi_n, t)| &= O(|t|^{-1-\epsilon}) \end{aligned}$$

for all multi-indices β, locally uniformly in $\xi_\perp = (0, \xi_2, \dots, \xi_n)$. (In particular in the one-dimensional case $Y \equiv 0$. In §3, Y is explicitly constructed.) Thus $X_D(t) = X(D_2, \dots, D_n, t)$.

Theorem 1.2 *Assume Hypotheses LR and SR. Then the modified wave operators $W_D^{\pm}$ exist and are complete and are isometries which intertwine H and H_0. Moreover the Møller wave operators $W^{\pm}$ exist if and only if $e^{iX(\xi_2, \dots, \xi_n, t)}$ converges in measure as $t \to \pm\infty$ on every compact subset of $\mathbf{R}^n$. Whenever $W^{\pm}$ exist, they are complete.*

Example. This continues the preceding example. Suppose for simplicity that b_1 and b_2 are nonzero and $\alpha \neq \beta$. Then the Møller wave operators (1.3) exist if and only if $\max\{\alpha, \beta\} + \epsilon > 1/2$ by Theorem 1.2. Ozawa [13] and Jensen-Ozawa [9] have already established a non-existence results for the Møller wave operators for a related class of potentials but by different methods.

Remark. In the case $n = 1$ the modifier depends only on time so that $e^{iX_D(t)} = e^{iX(t)}$ commutes with all operators. In particular, for any $u \in L^2(\mathbf{R}^n)$

$$|e^{-itH_0 - iX(t)} u(x)|^2 = |e^{-itH_0} u(x)|^2$$

which says that the position probability density of any free state is the same whether one uses the modified evolution or the usual free evolution. The same is true for the momentum probability density or any other observable in place of position or momentum. Therefore although the Møller wave operators $W^{\pm}$ do not exist the modified and free evolutions are indistinguishable by any quantum mechanical observable. It is therefore not surprising that in classical mechanics the usual wave operators exist as was observed by Jensen and Ozawa [9]. In general, for $n > 1$ the modifier is nontrivial. If however one further assumes

$$D^{\alpha}V_L(x) = O((1+|x|)^{-\alpha-\epsilon} \quad \text{for } |\alpha| \leq 1 \tag{1.7}$$

($\epsilon > 0$) then again one can replace $X(\xi, t)$ by a different modifier depending only on time (see Theorem 3.1 below) and which therefore cannot be observed. This last result is due to G.M. Graf [6] who assumed simply (1.7). Thus he requires less smoothness but more decay than here. He remarks that from the perspective of the Heisenberg picture of quantum mechanics there is no difference between quantum and classical mechanics in this setting. Graf uses Mourre's method.

In the remaining two Sections we outline the construction of $\theta^{\pm}$ for the proof of Theorem 1.1 (in §2). In §3 the proof of completeness in Theorem 1.2 is given; the remaining conclusions of Theorem 1.2 are standard and their proofs are only outlined.

2 Completeness of $W_D^{\pm}$.

In this Section we outline the construction of the operators $J^{\pm}$ of (1.5) or, more precisely, the phase terms $\theta^{\pm}$ as required for the proof of Theorem 1.1. In the process we indicate some key steps of the proof of Theorem 1.1 but our primary goal is to establish the properties of $\theta^{\pm}$ required for the proof of Theorem 1.2 in §3. A detailed proof of Theorem 1.1 is given in [15].

The construction of $\theta^{\pm}$ is as follows. It suffices to consider θ^{+}; the construction of θ^{-} is similar and in fact $\theta^{-}(x,\xi) = -\theta^{+}(x,-\xi)$. Choose $\chi_1 \in C^{\infty}(\mathbf{R})$ so that

$$\chi_1(x_1) = \begin{cases} 1 & \text{if } x_1 > 3 \\ 0 & \text{if } x_1 < 1 \end{cases} \tag{2.1}$$

The proof of Theorem 1.1 is based on the Enss method [4] in a two Hilbert space setting. One begins therefore with Cook's argument and so the key is to prove that the operator norm of $(d/dt)e^{itH}J^{+}e^{-itH_0}\chi_1(D_1)$ is an integrable function of $t > 1$, where $D_1 = -i\partial/\partial x_1$ so that $\chi_1(D_1)$ maps onto "outgoing states." The free evolution on outgoing states $e^{-itH_0}\chi_1(D_1)$ can be estimated