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Astérisque, tome 210 (1992), p. 355-384

<http://www.numdam.org/item?id=AST_1992__210__355_0>

RADIATION CONDITIONS AND SCATTERING THEORY FOR THREE-PARTICLE HAMILTONIANS

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1. INTRODUCTION

One of the main problems of scattering theory is a description of asymptotic behaviour of N interacting quantum particles for large times. The complete classification of all possible asymptotics (channels of scattering) is called asymptotic completeness. The final result can easily be formulated in physics terms. Two particles can either form a bound state or are asymptotically free. In case $N \geq 3$ a system of N particles can also be decomposed asymptotically into its subsystems (clusters). Particles of the same cluster form a bound state and different clusters do not interact with each other.

There are two essentially different approaches to a proof of asymptotic completeness for multiparticle ($N \geq 3$) quantum systems. The first of them, started by L. D. Faddeev [1], relies on the detailed study of a set of equations derived by him for the resolvent of the corresponding Hamiltonian. This approach was developped in [1] for the case of three particles and was further elaborated in [2, 3]. The attempts [4, 5] towards a straightforward generalization of Faddeev's method to an arbitrary number of particles meet with numerous difficulties. However, the results of [6] for weak interactions are quite elementary.

Another approach relies on the commutator method [7] of T. Kato. In the theory of N-particle scattering it was introduced by R. Lavine [8, 9] for repulsive potentials. A proof of asymptotic completeness in the general case is much more complicated and is due to I. Sigal and A. Soffer [10]. In the recent paper [11] G. M. Graf gave an accurate proof of asymptotic completeness in the time-dependent framework. The distinguishing feature of [11] is that all intermediary results are also purely time-dependent and most of them have a direct classical interpretation. Papers [10, 11] were to a large extent inspired by V. Enss (see e.g. [12]) who was the first to apply a time-dependent technique for the proof of asymptotic completeness.

The aim of the present paper is to give an elementary proof of asymptotic completeness (for the precise statement, see section 2) for three-particle Hamiltonians with short-range potentials which fits into the theory of smooth perturbations [7, 13]. Our approach admits a straightforward generalization to an arbitrary number of particles. This will be discussed elsewhere. Our proof of asymptotic completeness relies on new estimates which establish some kind of radiation conditions for three-particle systems. Compared to the limiting absorption principle (see below) radiation conditions-estimates give us an additional information on the asymptotic behaviour of a quantum system for large distances or large times. Limiting absorption principle suffices for a proof of asymptotic completeness in case of two-particle Hamiltonians with short-range potentials. However, radiation conditions-estimates are crucial in scattering for long-range potentials (see e.g. [14]), in scattering by unbounded obstacles [15, 16] and in scattering for anisotropically decreasing potentials [17]. In the latter paper the role of radiation conditions was also advocated for three-particle Hamiltonians. Our proof of radiation conditions-estimates hinges on the commutator method rather than the integration-by-parts machinery used in the two-particle case (see e.g. [14]).

Our interpretation of radiation conditions is, of course, different from the two-particle case. Before discussing their precise form let us introduce the generalized three-particle Hamiltonians. We consider the self-adjoint Schrödinger operator $H = -\Delta + V(x)$ in the Hilbert space $\mathcal{H} = L_2(\mathbb{R}^d)$. Suppose that some finite number α_0 of subspaces X^α of $X := \mathbb{R}^d$ is given and let x^α, x_α be the orthogonal projections of $x \in X$ on X^α and $X_\alpha = X \ominus X^\alpha$, respectively. We assume that

$$V(x) = \sum_{\alpha=1}^{\alpha_0} V^\alpha(x^\alpha), \tag{1.1}$$

where V^α are decreasing real functions of variables x^α. We prove asymptotic completeness under the assumption that V^α are short-range functions of x^α but many intermediary results (in particular, radiation conditions-estimates) are as well true for long-range potentials. Clearly, $V^\alpha(x^\alpha)$ tends to zero as $|x| \to \infty$ outside of any conical neighbourhood of X_α and $V^\alpha(x^\alpha)$ is constant on planes parallel to X_α. Due to this property the structure of the spectrum of H is much more complicated than in the two-particle case. Operators H considered here were introduced in [18] and are natural generalizations of N-particle Hamiltonians. We further assume that

$$X_\alpha \cap X_\beta = \{0\}, \quad \alpha \neq \beta, \tag{1.2}$$

so that regions where different V^α "live" have compact intersection (for potentials of compact support). For the Schrödinger operator this is true only for the case of three particles. Thus the assumption (1.2) distinguishes the three-particle problem.

Our proof of asymptotic completeness requires only the "angular part" of radiation conditions. Let $\langle\cdot,\cdot\rangle$ be the scalar product in the space $\mathbb{C}^d$ and let $\nabla^{(s)}$,

$$\nabla^{(s)}u(x) = \nabla u(x) - |x|^{-2}\langle\nabla u(x), x\rangle x, \tag{1.3}$$

be the projection of the gradient ∇ on the plane, orthogonal to x. Denote by χ_0 the characteristic function of any closed cone Γ_0 such that $\Gamma_0 \cap X_\alpha = \{0\}$ for all α. We prove that the operator

$$G_0 = \chi_0(|x|+1)^{-1/2}\nabla^{(s)} \tag{1.4}$$

is locally (away from thresholds and eigenvalues of H) H-smooth (in the sense of T. Kato – see e.g. [19]). In neighbourhoods of X_α we have only a weaker result. Namely, let ∇_{x_α} be the gradient in the variable x_α (i.e. $\nabla_{x_\alpha}u$ is the orthogonal projection of ∇u on X_α),

$$\nabla^{(s)}_{x_\alpha}u(x) = \nabla_{x_\alpha}u(x) - |x_\alpha|^{-2}\langle\nabla_{x_\alpha}u(x), x_\alpha\rangle x_\alpha \tag{1.5}$$

and let χ_α be the characteristic function of such a closed cone Γ_α that $\Gamma_\alpha \cap X_\beta = \{0\}$ for all $\beta \neq \alpha$. Then the operator

$$G_\alpha = \chi_\alpha(|x|+1)^{-1/2}\,\nabla^{(s)}_{x_\alpha} \tag{1.6}$$

is locally H-smooth. A definition of H-smoothness of the operators G_0 and G_α can be given either in terms of the resolvent of the operator H or of its unitary group $U(t) = \exp(-iHt)$. In both versions results are formulated as certain estimates which we call radiation conditions-estimates.

Our proof in section 3 of H-smoothness of the operators G_0 and G_α is based on consideration of the commutator $[H,M] := HM - MH$, where M is a self-adjoint first-order differential operator with *bounded* coefficients. We find an operator M such that $i[H,M]$ is essentially bounded from below by $G_0^*G_0$ and $G_\alpha^*G_\alpha$. Here we take into account that certain terms, those vanishing as $O(|x|^{-\rho}), \rho > 1$, at infinity, are negligible. This is a consequence of local H-smoothness of the operator $(|x|+1)^{-r}, r > 1/2$, (limiting absorption principle) which, in turn, is ensured by the Mourre estimate [20, 21, 22]. We emphasize that all our considerations are localized in energy.

The H-smoothness of the operators G_0 and G_α suffices for the proof in section 4 of existence of suitable wave operators (both "direct" and "inverse") with non-trivial identifications which are first-order differential operators. The sum of these identifications equals M, which allows us to find the asymptotics of $MU(t)f$ for large t. Since the limit $M^{\pm}$ as $t \to \pm\infty$ of the observable $U^*(t)\, M\, U(t)$ also exists, this gives the asymptotics of the function $U(t)f$ for f from the range of the operator $M^{\pm}$. Using again the Mourre estimate, we prove (also in section 4) that actually this range coincides with the whole absolutely continuous subspace of the Hamiltonian H. Finally, in section 5 we conclude our proof of asymptotic completeness.

2. BASIC NOTIONS OF SCATTERING THEORY

Let us briefly recall some basic definitions of the scattering theory. For a self-adjoint operator H in a Hilbert space $\mathcal{H}$ we introduce the following standard notation: $\mathcal{D}(H)$ is its domain; $\sigma(H)$ is its spectrum; $E(\Omega; H)$ is the spectral projection of H corresponding to a Borel set $\Omega \subset \mathbf{R}$; $\mathcal{H}^{(ac)}(H)$ is the absolutely continuous subspace of H; $P^{(ac)}(H)$ is the orthogonal projection on $\mathcal{H}^{(ac)}(H)$; $\mathcal{H}^{(p)}(H)$ is the subspace spanned by all eigenvectors of the operator H; $\sigma^{(p)}(H)$ is the spectrum of the restriction of H on $\mathcal{H}^{(p)}(H)$, i.e. $\sigma^{(p)}(H)$ is the closure of the set of all eigenvalues of H. Norms of vectors and operators in different spaces are denoted by the same symbol $\|\cdot\|$; I is always the identity operator; $\mathcal{B}$ and $\mathcal{K}_\infty$ are the classes of bounded and compact operators (in different spaces) respectively; C and c are positive constants whose precise values are of no importance; "$s-\lim$" means the strong operator limit. Note that

$$s - \lim_{|t|\to\infty} K \exp(-iHt) P^{(ac)}(H) = 0, \quad \text{if} \quad K \in \mathcal{K}_\infty. \tag{2.1}$$

Let K be H-bounded operator, acting from $\mathcal{H}$ into, possibly, another Hilbert space $\mathcal{H}'$. It is called H-smooth (in the sense of T. Kato) on a Borel set $\Omega \subset \mathbf{R}$ if for every $f = E(\Omega; H)f \in \mathcal{D}(H)$

$$\int_{-\infty}^{\infty} \|K \exp(-iHt) f\|^2 \, dt \leq C \|f\|^2.$$

Obviously, BK is H-smooth on Ω if K has this property and $B \in \mathcal{B}$.

Let now H_j, $j = 1, 2$, be a couple of self-adjoint operators and let J be a bounded operator in a Hilbert space $\mathcal{H}$. The wave operator for the pair H_1, H_2 and the "identification" J is defined by the relation

$$W^{\pm}(H_2, H_1; J) = s - \lim_{t\to\pm\infty} \exp(iH_2 t) J \exp(-iH_1 t) P^{(ac)}(H_1) \tag{2.2}$$

under the assumption that this limit exists. We emphasize that all definitions and considerations for " + " and " − " are independent of each other. It suffices to verify existence of the limit (2.2) on some set dense in $\mathcal{H}$. If the wave operator (2.2) exists, then the intertwining property

$$E_2(\Omega) W^{\pm}(H_2, H_1; J) = W^{\pm}(H_2, H_1; J) E_1(\Omega) \tag{2.3}$$