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ON NONLINEAR SCATTERING OF STATES WHICH ARE CLOSE TO A SOLITON

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1 Solitons

Consider the nonlinear Schroedinger equation

$$i\psi_t = -\psi_{xx} + F(|\psi|^2)\psi, \psi = \psi(x,t) \in \mathbf{C}, \tag{1.1}$$

$x, t \in \mathbf{R}$. Assume that
i)F is a given smooth ($\in C^\infty$) real function bounded from below,
ii)the point $\xi = 0$ is a (sufficiently strong) root of the function F:

$$F(\xi) = F_1\xi^p(1 + O(\xi)), p > 0. \tag{1.2}$$

Further consider the function

$$U(\phi, \alpha) = -\frac{1}{8}\alpha^2\phi^2 - \frac{1}{2}\int_0^{\phi^2} F(\xi)d\xi. \tag{1.3}$$

If $\alpha \neq 0$ this function is negative for sufficiently small ϕ. The next assumption on F will be given in a sligtly implicit, but absolutely elementary form:
iii)for α from some interval, $\alpha \in A \subset \mathbf{R}_+$, the function $\phi \to U(\phi, \alpha)$ has a positive root; if $\phi_0(= \phi_0(\alpha))$ is the smallest positive root then $U_\phi(\phi_0, \alpha) > 0$.

Under all these assumptions there exists the unique even positive solution $y \to \phi(y)$ of the equation

$$\phi_{yy} = -U_\phi = \frac{1}{4}\alpha^2\phi + F(\phi^2)\phi \tag{1.4}$$

vanishing at infinity. More precisely

$$\phi = \phi(y|\alpha) \sim \phi_\infty exp(-\frac{1}{2}\alpha|y|), y \to \infty. \tag{1.5}$$

The following functions of x can be called the *soliton states*:

$$w(x|\sigma) = exp(-i\beta + i\frac{1}{2}vx)\phi(x - b|\alpha), \tag{1.6}$$

here

$$\sigma = (\beta, \omega, b, v), \omega = \frac{1}{4}(v^2 - \alpha^2), \tag{1.7}$$

$\beta, \omega, b, v \in \mathbf{R}, \alpha \in A$. The set of the allowable σ will be denoted by Σ. If σ is a solution of the Hamiltonian system:

$$\beta' = \omega, \omega' = 0, b' = v, v' = 0. \tag{1.8}$$

the function $w(x|\sigma(t))$ is a solution of the equation (1.1) called the *soliton*.

2 The linearization of equation (1.1)

Consider the linearization of the equation (1.1) on the soliton $w(x|\sigma(t))$:

$$i\chi_t = -\chi_{xx} + F(|w|^2)\chi + F'(|w|^2)w(\bar{w}\chi + w\bar{\chi}). \tag{2.1}$$

Instead of χ introduce the function f:

$$\chi(x,t) = exp(i\Phi)f(y,t), \Phi = -\beta(t) + \frac{1}{2}vx, y = x - b(t). \tag{2.2}$$

The function f obeys the following equation:

$$if_t = L(\alpha)f, \tag{2.3}$$

where

$$L(\alpha)f = -f_{yy} + \frac{1}{4}\alpha^2 f + F(\phi^2)f + F'(\phi^2)\phi^2(f + \bar{f}), \tag{2.4}$$

$\phi = \phi(y|\alpha)$. Equation (2.3) is only a real-linear equation. Introduce its complexification:

$$i\vec{f_t} = H(\alpha)\vec{f}, \vec{f} = \begin{pmatrix} f \\ \bar{f} \end{pmatrix}, \tag{2.5}$$

$$H(\alpha) = H_0(\alpha) + V(\alpha), H_0(\alpha) = (-\partial_y^2 + \frac{1}{4}\alpha^2)\sigma_3, \tag{2.6}$$

$$V(\alpha) = \left[F(\phi^2) + F'(\phi^2)\phi^2\right]\sigma_3 + iF'(\phi^2)\phi^2\sigma_2, \tag{2.7}$$

σ_2, σ_3 are the standard Pauli matrices:

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.8}$$

3 Properties of the operator $H(\alpha)$

The operator $H(\alpha)$ can be treated as a linear operator in $\mathbf{L}_2(\mathbf{R} \to \mathbf{C}^2)$. Define it on the domain where $H_0(\alpha)$ is self-adjoint. It possesses the properties:

$$\sigma_3 H = H^* \sigma_3, \sigma_2 H = -H^* \sigma_2, \sigma_1 H = -H \sigma_1. \tag{3.1}$$

As a result the spectrum of H is invariant with respect to the following transformations: $E \to \bar{E}, E \to -E$.

The continuous spectrum consists of two half-axis $[E_0, \infty)$ and $(-\infty, -E_0]$, $E_0 = \frac{1}{4}\alpha^2$. Its multiplicity is equal to 2.

Owing to the exponential decay of the potential term $V(\alpha)$ at infinity the discrete spectrum of $H(\alpha)$ contains only a finite number of eigenvalues and the corresponding *root subspaces* have only finite dimension.

The point $E = 0$ is always a point of the discrete spectrum. One can indicate two *eigenfunctions*

$$\vec{\xi_1} = \begin{pmatrix} u_1 \\ \bar{u}_1 \end{pmatrix}, \vec{\xi_3} = \begin{pmatrix} u_3 \\ \bar{u}_3 \end{pmatrix}, \tag{3.2}$$

where

$$u_1 = -i\phi(y|\alpha), u_3 = -\phi_y, \tag{3.3}$$

and two *adjoint functions*:

$$\vec{\xi_2} = \begin{pmatrix} u_2 \\ \bar{u}_2 \end{pmatrix}, \vec{\xi_4} = \begin{pmatrix} u_4 \\ \bar{u}_4 \end{pmatrix}, \tag{3.4}$$

where

$$u_2 = -\frac{2}{\alpha}\phi_\alpha, u_4 = \frac{i}{2}y\phi. \tag{3.5}$$

They obey the relations:

$$H\vec{\xi_1} = H\vec{\xi_3} = 0, H\vec{\xi_2} = i\vec{\xi_1}, H\vec{\xi_4} = i\vec{\xi_3}. \tag{3.6}$$

Actually, the spectrum of $H(\alpha)$ can lie only in the real axis and in the imaginary axis of the E-plane, see [Wc2], for example. It is known also that the spectrum of $H(\alpha)$ is real and the root subspace corresponding to the point $E = 0$ is generated by the vectors $\vec{\xi_1}, \vec{\xi_2}, \vec{\xi_3}, \vec{\xi_4}$ if and only if

$$\partial_\alpha \|\phi\|^2 > 0. \tag{3.7}$$

Consider the resolvent $R(E) = (H - E)^{-1}$. Its kernel $R(y, y'|E)$ is an analytic function in the *extended* E-plane: it admits an analytic continuation through the continuous spectrum as a meromorphic function. The resolvent kernel goes to infinity when E tends to the branch points $\mp E_0$ if the equation $H(\alpha)\psi = \mp E_0 \psi$, treated as a differential equation, has nontrivial solutions bounded at infinity. In this case the points $\mp E_0$ will be called *resonances*.

4 Nonlinear equation

Consider the Cauchy problem for equation (1.1) with the initial data

$$\psi(x,0) = \psi_0(x), \tag{4.1}$$

where $\psi_0 \in H^1, H^1$ is the standard Sobolev space with the norm:

$$\|f\|_{H^1}^2 = \|f\|_2^2 + \|f'\|_2^2. \tag{4.2}$$

The problem has a solution $\psi = \psi(x,t)$ which belongs to H^1 with respect to x for each t, moreover $\psi \in C(\mathbf{R} \to H^1)$. Any such solution ψ obeys two *conservation laws*:

$$\int |\psi(x,t)|^2 dx = const, \int \left[|\psi_x(x,t)|^2 + U(|\psi(x,t)|)\right] dx = const, \tag{4.3}$$

where U is the function (1.3). The second formula (4.3) leads to the following estimate:

$$\|\psi(\cdot,t)\|_{H^1} \le c\,(\|\psi_0\|_{H^1})\,\|\psi_0\|_{H^1}, \tag{4.4}$$

here $c = \mathbf{R}_+ \to \mathbf{R}_+$ is a smooth function. If in addition ψ_0 has the finite norm: $\|(1+|x|)\psi_0\|_2 < \infty$, the solution ψ also has the finite , but growing in time, similar norm:

$$\|(1+|x|)\psi(x,t)\|_2 \le c(\|\psi_0\|_{H^1})\left[\|(1+|x|)\psi\|_2 + t\|\psi_0\|_{H^1}\right]. \tag{4.5}$$

5 Theorem

Let $\sigma_0 = (\beta_0, \omega_0, b_0, v_0) \in \Sigma, \omega_0 = \frac{1}{4}(v_0^2 - \alpha_0^2)$. Consider the Cauchy problem for equation (1.1) with the initial data:

$$\psi_0(x) = w(x|\sigma_0) + \chi_0(x). \tag{5.1}$$

Our aim is to describe the asymptotic behavior of the solution ψ as $t \to \infty$. Assume that:

T_1) the norm

$$N = \|(1+x^2)\chi_0\|_2 + \|\chi_0'\|_2 \tag{5.2}$$

is sufficiently small;

T_2) $E = 0$ is the only point of the discrete spectrum of $H(\alpha_0)$ and the dimension of the corresponding root subspace is equal to 4;