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On the Spectrum of Gauge-Periodic Elliptic Operators

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1. Introduction

This note presents an extension of the results in [1] concerning the spectrum of symmetric elliptic operators on complete noncompact Riemannian manifolds. Thus consider a complete Riemannian manifold, M, of dimension m, with a properly discontinuous action of a discrete group, Γ, of isometries; we assume that the orbit space is compact. Moreover, let $E \to M$ be a hermitian vector bundle with a unitary representation

$$U : \Gamma \to L^2(E)\,. \tag{1.1a}$$

More precisely, we assume that Γ acts unitarily on E, via γ_*, and put

$$U_\gamma f(p) := \gamma_* f(\gamma^{-1}(p))\,. \tag{1.1b}$$

Thus each U_γ maps $C_0^\infty(E)$ to itself. Finally, let D be a symmetric elliptic differential operator on $C_0^\infty(E)$. In [1] we have assumed that D is, in addition, periodic in the sense that it commutes with all U_γ on $C_0^\infty(E)$. Now we bring in a second unitary representation, the *gauge*,

$$\begin{aligned} &V : \Gamma \to C^\infty(\operatorname{End} E)\,,\\ &V_\gamma \mid E_p \text{ is unitary for all } \gamma \in \Gamma,\, p \in M, \end{aligned} \tag{1.2}$$

which induces a unitary representation on $L^2(E)$. This representation will also be denoted by V. In general,

$$W_\gamma := V_\gamma U_\gamma \tag{1.3}$$

will not define a representation any more, since $[V_{\gamma_1}, U_{\gamma_2}]$ maybe nonzero. But frequently we have a good substitute namely

$$U_{\gamma_1} V_{\gamma_2} = X(\gamma_1, \gamma_2) V_{\gamma_2} U_{\gamma_1}\,, \tag{1.4a}$$

where

$$X(\gamma_1,\gamma_2) \text{ is in } C^\infty(\operatorname{End} E)\text{, unitary on each fiber, and a character of } \Gamma \text{ in each variable separately.} \tag{1.4b}$$

Moreover, we want that

$$X(\gamma,\gamma) = 1 \qquad \text{for all } \gamma \in \Gamma\,. \tag{1.4c}$$

The operator D is called *gauge-periodic* if

$$[W_\gamma, D] = 0 \quad \text{ on } C_0^\infty(E)\,. \tag{1.5}$$

The periodic case is obviously contained with V, X trivial. An interesting example with nontrivial gauge is provided by the Schrödinger operator with constant magnetic field in $\mathbb{R}^2$. This will be our main application which we deal with in greater detail below.

Assuming (1.5) we associate a C^*-algebra with D as follows. Fix a fundamental domain, $\mathcal{D}$, for Γ and introduce the isometry

$$\begin{aligned} &\Phi : L^2(E) \to L^2(\Gamma, L^2(E \mid \mathcal{D}))\,,\\ &\Phi f(\gamma) := r_{\mathcal{D}} \circ W_\gamma(f)\,, \end{aligned} \tag{1.6}$$

where $r_{\mathcal{D}}$ denotes restriction $L^2(E) \to L^2(E \mid \mathcal{D}) =: H$. Let R_γ, L_γ be right translation by γ and left translation by γ^{-1} in $L^2(\Gamma)$, respectively, and define the unitary operator X_γ in $L^2(\Gamma)$ for $\gamma \in \Gamma$ by

$$X_\gamma \sigma(\delta) := X(\delta,\gamma)\sigma(\delta)\,. \tag{1.7}$$

Then it is easy to compute that

$$\tilde{R}_\gamma := \Phi W_\gamma \Phi^{-1} = X_\gamma R_\gamma \otimes I\,. \tag{1.8}$$

Since X is a bicharacter, it is also readily seen that

$$[X_{\gamma_1} L_{\gamma_1} \otimes I, \tilde{R}_{\gamma_2}] = 0 \qquad \text{for all } \gamma_1,\gamma_2 \in \Gamma\,. \tag{1.9}$$

We will see that this is satisfied in our main example (and probably in many other cases). Then we abbreviate $\tilde{L}_\gamma =: X_\gamma L_\gamma$ and introduce the C^*-algebra $\mathcal{C}_W(\Gamma)$ which is generated by $(\tilde{L}_\gamma)_{\gamma\in\Gamma}$ in $\mathcal{L}(L^2(\Gamma))$. With $\mathcal{K} = \mathcal{K}(H)$, the ideal of compact operators on $H = L^2(E \mid \mathcal{D})$, we introduce, as in [1],

$$\mathcal{C}_W(\Gamma,\mathcal{K}) := \mathcal{C}_W(\Gamma) \otimes \mathcal{K}\,. \tag{1.10}$$

On this algebra we can again define a natural trace tr_Γ (to be described in Sec. 3), such that all spectral projections of D have a finite trace. We say that $\mathcal{C}_W(\Gamma, \mathcal{K})$ has the *Kadison property* if there is a constant $C > 0$ such that

$$\mathrm{tr}_\Gamma P \geq C\,, \tag{1.11}$$

for all nonzero orthogonal projections $P \in \mathcal{C}_W(\Gamma, \mathcal{K})$. The largest constant in (1.11) will be called the *Kadison constant* of $\mathcal{C}_W(\Gamma, \mathcal{K})$, to be denoted $C_W(\Gamma)$.

We can show that D has a unique self-adjoint extension, $\bar{D}$, with spectral resolution

$$\bar{D} = \int_{-\infty}^{+\infty} \lambda dE_\lambda\,.$$

Quite analogously to [1] we then obtain

Theorem 1 *If $\lambda_1 > \lambda_2 \in \mathbb{R}\backslash\mathrm{spec}\,\bar{D}$ then $E_{\lambda_1} - E_{\lambda_2} \in \mathcal{C}_W(\Gamma, \mathcal{K})$. If $\mathcal{C}_W(\Gamma)$ has the Kadison property then the spectrum of $\bar{D}$ has band structure in the sense that the intersection of the resolvent set with any compact interval of real numbers has finitely many components.*

As noted in [1], the proof of Theorem 1 gives some quantitive information which we exploit in connection with the magnetic Schrödinger operator in $\mathbb{R}^2$. Recall that this operator is defined on $C_0^\infty(\mathbb{R}^2)$ by

$$D_A := \sum_{i=1}^{2}\Big(\frac{1}{\sqrt{-1}}\frac{\partial}{\partial x_i} + a_i\Big)^2 + v\,, \tag{1.12}$$

where $a_i, v \in C^\infty(\mathbb{R}^2)$. The magnetic field is assumed to be constant,

$$b(x) := \Big(\frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2}\Big)(x) \equiv b(0) =: b\,,$$

and we assume moreover that v is $\mathbb{Z}^2$-periodic. b is also equal to the magnetic flux over a unit cell,

$$b = \int_{0 \leq x_1, x_2 \leq 1} b(x_1, x_2) dx_1 dx_2 =: 2\pi\theta\,. \tag{1.13}$$

This operator fits into our framework as follows. Since the magnetic field is constant we may assume that

$$a_1(x) = bx_2/2\,, \quad a_2(x) = -bx_1/2\,.$$

With ω the standard symplectic form in $\mathbb{R}^2$, we define for $z \in \mathbb{Z}^2$

$$\begin{aligned} U_z f(x) &:= f(x-z)\,, \\ V_z f(x) &:= e^{\sqrt{-1}\, b/2\, \omega(x,z)} f(x)\,. \end{aligned} \tag{1.14}$$

Then it follows that

$$X(z_1, z_2) = e^{\sqrt{-1}\, b/2\, \omega(z_1,z_2)} \tag{1.15}$$

and

$$\tilde{L}_{z_1} \tilde{L}_{z_2} = e^{\sqrt{-1}\, b\, \omega(z_2,z_1)} \tilde{L}_{z_2} \tilde{L}_{z_1}\,. \tag{1.16}$$

Now we regard the quantity b as a parameter restricted by $|b| \le C_1$, say. It is known that the precise band structure of spec $\bar{D}_A$ in a given interval $[\lambda_1, \lambda_2]$ depends rather subtly on the arithmetic nature of θ in (1.13). We will prove

Theorem 2 *Assume that $\theta = p/q \in \mathbb{Q}$ with $(p,q) = 1$, and that $\lambda_1 > \lambda_2 \in \mathbb{R} \setminus$ spec $\bar{D}_A$. There is a constant C depending only on C_1, λ_1, λ_2, and v such that*

$$G(D_A, \lambda_1, \lambda_2) := \sharp\{ \text{ gaps in spec } \bar{D}_A \cap [\lambda_2, \lambda_1]\}$$

satisfies the estimate

$$G(D_A, \lambda_1, \lambda_2) \le C(C_1, \lambda_1, \lambda_2, v)\, q\,. \tag{1.17}$$

The proof of this result uses the fact that the Kadison constant of $\mathcal{C}_W(\Gamma)$ satisfies $C_W(\Gamma) \ge q^{-1}$. This degeneration then allows the possible development of Cantor structures if θ approaches irrational numbers. It has been shown in [3] that, for suitable v, G also has a similar lower bound. Crucial for this result was a thorough study of Harper's equation, a discrete approximation to D_A. Using only the structure of the rotation algebras (which are brought in by (1.16)) it has been shown in [2] that the maximum number of gaps is realized by Harper's operator. One might thus hope that our approach, which links all gauge-periodic operators with the rotation algebra, opens a way to bypass the discrete approximation and to establish directly that "sufficiently complicated" operators in the rational rotation algebra will indeed have the maximum number of gaps. Of course, this need not be so for every operator as illustrated by the case $v = 0$. Since $C_W(\Gamma) = 0$ for irrational θ, we also see that for a vanishing Kadison constant no general conclusion concerning the structure of the spectrum is possible.

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