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Graded C^*-Algebras and Many-Body Perturbation Theory: II. The Mourre Estimate

Anne Boutet de Monvel-Berthier and Vladimir Georgescu [1]

1. Introduction

We have introduced in [BG 1,2] the notion of graded C^*-algebra with the purpose of obtaining a natural framework for the description and study of hamiltonians with a many-channel structure. If H is a self-adjoint operator in a Hilbert space $\mathscr{H}$, the expression "H has a many-channel structure" is not mathematically well defined, although in examples of physical interest the meaning is rather obvious. Spectral theory alone is not enough in order to decide whether H is a many-channel hamiltonian or not. Usually the distinction is acquired with the help of scattering theory through the introduction of the channel wave operators. However, there are results (like the HVZ theorem which describes the essential spectrum of a N-body hamiltonian in terms of the spectra of the subsystems) which are outside the scope of scattering theory but should belong to a general theory of "many-channel hamiltonians". Our proposal in [BG 1,2] was to define the many-channel character of a self-adjoint operator H by its affiliation to a C^*-algebra provided with a graduation which allows one to describe a "subsystem structure" for the system whose hamiltonian is H. From our point of view, the main object associated to the physical system is a graded C^*-algebra, the possible dynamics are given by self-adjoint operators H affiliated to it, and we are interested in assertions independent of the explicit form of H.

Our purpose here is to show that the Mourre estimate fits very nicely in such a framework. Given two self-adjoint operators H, A such that the commutator [H,A] is a continuous sesquilinear form on D(H), we associate to them a function $\rho:\mathbb{R}\to]-\infty,+\infty]$ in terms of which the property of A of being locally conjugated to H is easily described. If the action of the unitary group associated to A is compatible in some sense with the grading of the C^*-algebra and if this algebra has a property which we call reducibility, then the ρ-function associated to H can be estimated in terms of the ρ-functions associated to "sub-hamiltonians". Our arguments are inspired from those of Froese and Herbst [FH], but the main point here is that the explicit form of H is never used, but only its affiliation to the algebra. In particular, in the N-body case H could be of the form described in

[1] Lecture delivered by V. Georgescu

Proposition 7 of [BG 2] (see also section 2 below; this class is more general than the class of dispersive hamiltonians of [D2] and [G]) or it could be a hamiltonian with hard core interactions (this situation is treated in a joint work with A.Soffer, paper in preparation). We shall explicitely calculate the ρ-function (and so get the result of [PSS] and [FH]) for Agmon hamiltonians using theorem 3.4 which gives the ρ-function of an operator $H = H_1 \otimes 1 + 1 \otimes H_2$ in terms of those of H_j assuming that A is similarly decomposable. Theorems 3.4 and 4.4 are, technically speaking the main results of this paper, the applications to hamiltonians affiliated to the N-body C^*-graded algebra, being only an example (in this context theorem 2.1 being important)

In the rest of this section we shall recall the framework introduced in [BG1,2]. Some more specific properties of what we call the N-body C^*-graded algebra are studied in section 2. In section 3 we introduce in a more general setting the ρ-functions (which are more systematically studied in [ABG 2]) and prove the first important result, formula (3.8). Finally, in section 4 we define the reducible algebras and show how a Mourre estimate is proved for hamiltonians affiliated to such algebras.

We recall now the definition of a C^*-graded algebra as introduced in [BG1,2]. Let $\mathscr{A}$ be a C^*-algebra and $\mathscr{L}$ a finite lattice, i.e. a finite partially ordered set such that the upper bound $Y \vee Z$ and the lower bound $Y \wedge Z$ of each pair $Y,Z \in \mathscr{L}$ exists. We shall denote $\mathbf{O}$ (resp. X) the least (resp. the biggest) element of $\mathscr{L}$. We say that *$\mathscr{A}$ is a $\mathscr{L}$-graded C^*-algebra* if a family $\{\mathscr{A}(Y)\}_{Y \in \mathscr{L}}$ of C^*-subalgebras of $\mathscr{A}$ is given such that

(i) $\mathscr{A} = \Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$, the sum being direct (as linear spaces);

(ii) $\mathscr{A}(Y)\mathscr{A}(Z) \subset \mathscr{A}(Y \vee Z)$ for all $Y,Z \in \mathscr{L}$.

One can introduce such a notion for infinite $\mathscr{L}$ also (then $\Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$ is only dense in $\mathscr{A}$) and an interesting example of such an object will appear in the next section.

We can put in evidence a *filtration* of $\mathscr{A}$ by a family $\{\mathscr{A}_Y\}_{Y \in \mathscr{L}}$ of C^*-subalgebras by defining $\mathscr{A}_Y = \Sigma\{\mathscr{A}(Z) \mid Z \leq Y\}$. Then $\mathscr{A}_Y \subset \mathscr{A}_Z$ if $Y \leq Z$ and $\mathscr{A}_X = \mathscr{A}$. If we denote $\mathscr{L}(Y) = \{Z \in \mathscr{L} \mid Z \leq Y\}$, then $\mathscr{L}(Y)$ is a finite lattice also and $\mathscr{A}_Y$ is a $\mathscr{L}(Y)$-graded C^*-algebra in a canonical way. Finally, observe that $\mathscr{A}(X)$ is a $*$-ideal in $\mathscr{A}$ (so $\mathscr{A}(Y)$ is a $*$-ideal in $\mathscr{A}_Y$), and if we denote $\mathscr{B}_Y = \Sigma\{\mathscr{A}(Z) \mid Z \nleq Y\}$, then $\{\mathscr{B}_Y\}_{Y \in \mathscr{L}}$ is a decreasing family of closed $*$-ideals in $\mathscr{A}$ such that $\mathscr{A} = \mathscr{A}_Y + \mathscr{B}_Y$ (algebraic direct sum) for all $Y \in \mathscr{L}$.

For each $Y \in \mathscr{L}$ we shall denote $\mathscr{P}(Y)$, $\mathscr{P}_Y$ the projection operators of $\mathscr{A}$ onto $\mathscr{A}(Y)$, resp. $\mathscr{A}_Y$, associated to the direct sum decompositions $\mathscr{A} = \Sigma\{\mathscr{A}(Y) \mid Y \in \mathscr{L}\}$

resp. $\mathscr{A}=\mathscr{A}_Y+\mathscr{B}_Y$. More precisely, if $S\in\mathscr{A}$, then one can write it in a unique way as a sum $S=\Sigma\{S(Y)\mid Y\in\mathscr{L}\}$ with $S(Y)\in\mathscr{A}(Y)$. Then $\mathscr{P}(Y)(S)=S(Y)$. Obviously $\mathscr{P}_Y=\Sigma\{\mathscr{P}(Z)\mid Z\leq Y\}$, which is equivalent to $\mathscr{P}(Y)=\Sigma\{\mathscr{P}_Z\,\mu(Z,Y)\mid Z\leq Y\}$, where $\mu:\mathscr{L}\times\mathscr{L}\to\mathbb{Z}$ is the Möbius function of $\mathscr{L}$. Clearly each $\mathscr{P}(Y):\mathscr{A}\to\mathscr{A}$ is a linear, continuous projection (i.e.$\mathscr{P}(Y)^2=\mathscr{P}(Y)$) which commutes with the involution. But the main point is that $\mathscr{P}_Y:\mathscr{A}\to\mathscr{A}$ *is a linear, continuous projection which is also a* $*$*-homomorphism of* $\mathscr{A}$ *onto* $\mathscr{A}_Y$. In particular, if $S\in\mathscr{A}$ is a normal element and f is a complex continuous function on the spectrum of S (which vanishes at zero if $\mathscr{A}$ has not unit) then $\mathscr{P}_Y(f(S))=f(\mathscr{P}_Y(S))$. Observe that $\mathscr{B}_Y=\ker\mathscr{P}_Y$, which gives a new proof of the fact that $\mathscr{B}_Y$ is a closed $*$-ideal in $\mathscr{A}$.

Let $\mathscr{A}$ be an arbitrary C^*-algebra realised on a Hilbert space $\mathscr{H}$ (i.e. $\mathscr{A}$ is a C^*-subalgebra of $B(\mathscr{H})$, the space of bounded linear operators in $\mathscr{H}$) and H a self-adjoint operator in $\mathscr{H}$. Denote $C_\infty(\mathbb{R})$ the abelian C^*-algebra of complex continuous functions on $\mathbb{R}$ which tend to zero at infinity (with the sup norm). Then $(\lambda-H)^{-1}\in\mathscr{A}$ for some complex λ if and only if $f(H)\in\mathscr{A}$ for all $f\in C_\infty(\mathbb{R})$. If this is fulfilled, we shall say that H *is affiliated to* $\mathscr{A}$. In some applications it is useful to work with *self-adjoint but non-densely defined operators in* $\mathscr{H}$. By this we mean that a closed subspace $\mathscr{K}$ of $\mathscr{H}$ and a self-adjoint densely defined operator H in $\mathscr{K}$ are given (so $\mathscr{K}$ is the closure of the domain of H in $\mathscr{H}$; think, formally, that $H=\infty$ on $\mathscr{H}\ominus\mathscr{K}$). Let then $R(\lambda)=(\lambda-H)^{-1}$ on $\mathscr{K}$ and $R(\lambda)=0$ on $\mathscr{H}\ominus\mathscr{K}$, for $\lambda\in\mathbb{C}\setminus\mathbb{R}$. Clearly, the family $\{R(\lambda)\mid\lambda\in\mathbb{C}\setminus\mathbb{R}\}$ of bounded operators in $\mathscr{H}$ is a pseudo-resolvent, i.e. $R(\lambda)^*=R(\lambda^*)$ and $R(\lambda_1)-R(\lambda_2)=(\lambda_2-\lambda_1)R(\lambda_1)R(\lambda_2)$. In fact, as shown in [HP], there is a bijective correspondence between (not necessarily densely defined) self-adjoint operators in $\mathscr{H}$ and pseudo-resolvents on $\mathscr{H}$ (or spectral measures E such that $E(\mathbb{R})\neq 1$). Using Stone-Weierstrass theorem, it is trivial to establish a bijective correspondence between pseudo-resolvents and $*$-homomorphisms $\phi:C_\infty(\mathbb{R})\to B(\mathscr{H})$ (put $R(\lambda)=\phi(r_\lambda)$ where $r_\lambda(x)=(\lambda-x)^{-1}$). Clearly $\phi(f)|_{\mathscr{K}}=f(H)$ and $\phi(f)|_{\mathscr{H}\ominus\mathscr{K}}=0$.

As a conclusion of this discussion, if $\mathscr{A}$ is an arbitrary C^*-algebra, a $*$-homomorphism $\phi:C_\infty(\mathbb{R})\to\mathscr{A}$ will be called *self-adjoint operator affiliated to* $\mathscr{A}$. As above, to give ϕ is equivalent to giving a pseudo-resolvent $\{R(\lambda)\mid\lambda\in\mathbb{C}\setminus\mathbb{R}\}$ with $R(\lambda)\in\mathscr{A}$. We shall use in such a case a symbol H and denote $\phi(f)=f(H)$ for $f\in C_\infty(\mathbb{R})$ and $R(\lambda)=(\lambda-H)^{-1}$. When $\mathscr{A}$ is realised in a Hilbert space $\mathscr{H}$, then H is realised as a (non-densely defined in general) self-adjoint operator in $\mathscr{H}$. If $\mathscr{A}_1$ is another C^*-algebra and $\mathscr{P}:\mathscr{A}\to\mathscr{A}_1$ is a $*$-homomorphism then $\mathscr{P}\phi:C_\infty(\mathbb{R})\to\mathscr{A}_1$ is a $*$-homomorphism which defines a self-adjoint operator H_1 affiliated to $\mathscr{A}_1$. We shall denote $H_1=\mathscr{P}(H)$.

Let us go back now to our $\mathscr{L}$-graded C^*-algebra $\mathscr{A}$. For each self-adjoint operator H affiliated to $\mathscr{A}$ and each $Y\in\mathscr{L}$ we may consider the self-adjoint operator H_Y affiliated to $\mathscr{A}_Y$ defined by $H_Y=\mathscr{P}_Y(H)$ (i.e. $f(H_Y)=\mathscr{P}_Y(f(H))$ for all $f\in C_\infty(\mathbb{R})$). Observe that $H_X=H$. If H is just an element of $\mathscr{A}$, then $H_Y=\mathscr{P}_Y(H)$ is just the projection of H onto $\mathscr{A}_Y$. If $\mathscr{A}$ is realised on a Hilbert space $\mathscr{H}$ and H is the hamiltonian of a system (i.e. e^{-iHt} describes the time evolution of the system), then the H_Y's will be called sub-hamiltonians (they describe the evolution of the system when parts of the interaction have been suppressed). Observe that each H_Y (and $H=H_X$) has its own domain $D(H_Y)$ which is not dense in $\mathscr{H}$ in general. In the many-body case with hard-core interactions, D(H) is not dense, $D(H_{\mathbf{O}})$ is dense and $D(H_Y)$ for $Y\neq\mathbf{O}$, X is sometimes dense and sometimes not. If H_Y is densely defined for all Y, we shall say that the densely defined self-adjoint operator H in $\mathscr{H}$ is $\mathscr{L}$-affiliated to $\mathscr{A}$. Such operators are easy to construct using the following criterion. Let $H_{\mathbf{O}}=H(\mathbf{O})$ by a densely defined self-adjoint operator in $\mathscr{H}$ affiliated to $\mathscr{A}_{\mathbf{O}}=\mathscr{A}(\mathbf{O})$. For each $Y\neq\mathbf{O}$, let H(Y) be a symmetric, $H_{\mathbf{O}}$-bounded operator in $\mathscr{H}$ with relative bound zero and such that $H(Y)(H_{\mathbf{O}}+i)^{-1}\in\mathscr{A}(Y)$. Then $H=\Sigma\{H(Y)\,|\,Y\in\mathscr{L}\}$ is self-adjoint and $\mathscr{L}$-affiliated to $\mathscr{A}$. Moreover, for all $Y\in\mathscr{L}$, we have $H_Y=\mathscr{P}_Y(H)=\Sigma\{H(Z)\,|\,Z\leq Y\}$. If $H_{\mathbf{O}}$ is bounded below, then it is enough that H(Y) be $H_{\mathbf{O}}$-form bounded with relative bound zero and for c large enough $(H_{\mathbf{O}}+c)^{-1/2}H(Y)(H_{\mathbf{O}}+c)^{-1/2}\in\mathscr{A}(Y)$.

We stop here this accumulation of definitions. In [BG 2] these notions are used in the spectral theory of N-body systems. For example, we show that the Weinberg-Van Winter equation and the HVZ theorem are very natural in this framework (both the statements and the proofs).

2. The N-body Algebra

In this section we shall describe some important properties of a graded C^*-algebra canonically associated to an Euclidean space (in place of the usual N-body formalism, we prefer to work in the geometrical setting first considered by Agmon, Froese and Herbst and systematically developed in [ABG 1]).

Let E be an Euclidean space (finite dimensional real Hilbert space). We provide it with the unique translation invariant Borel measure such that the volume