Astérisque

VICTOR GUILLEMIN

The homogeneous Monge-Ampère equation on a pseudoconvex domain

Astérisque, tome 210 (1992), p. 97-113

<http://www.numdam.org/item?id=AST_1992__210__97_0>

The homogeneous Monge-Ampere equation on a pseudoconvex domain

Victor Guillemin*

§1. Introduction

Let X be a compact complex n-dimensional manifold with a smooth strictly-pseudoconvex boundary. Without loss of generality one can assume that X sits inside an open complex manifold, Z. A smooth function, $\phi : Z \longrightarrow \mathbb{R}$, is a *defining function* of X if it has the property:

$$\phi(p) \leq 1 \Longleftrightarrow p \in X$$

and if it has no critical points on the boundary. There are an infinity of different ways of choosing such a defining function, and it is a problem of considerable interest in the theory of pseudoconvex domains to find ways of making *canonical* choices. Jack Lee proved a result in his thesis which sheds some light on this problem: Suppose all the data above are real-analytic. Let S be the boundary of X and let $\Gamma \longrightarrow S$ be the bundle of outward-pointing conormal vectors to S. Given a real-analytic section, $\mu : S \longrightarrow \Gamma$, Lee proved that there exists a unique real-analytic defining function, ϕ, which satisfies the boundary condition, $d\phi = \mu$ on S and satisfies the homogeneous Monge-Ampere equation

$$(\bar{\partial}\partial\phi)^n = 0 \tag{1.1}$$

on a neighborhood of S.* One of the aims of this paper is to give a new proof of this result. This proof is similar to a proof that Matt Stenzel and I gave of an existence theorem for Monge-Ampere with a different set of boundary conditions in $[GS]_1$. I will give a brief description of this proof below; however, first I want to describe the other main result of this paper. Let X be a compact Riemannian manifold. Suppose that X is real-analytic, and suppose that $f : X \longrightarrow \mathbb{R}$ is a real-analytic function. Several years ago Boutet de Monvel proved the following surprising result:

Theorem. *[B] The following are equivalent*

1. *f can be extended holomorphically to a Grauert tube of radius r about X.*
2. *The wave equation*

$$\frac{\partial u}{\partial t} = \sqrt{\Delta} u \,, \qquad u(x,0) = f(x)$$

*Supported by NSF grant DMS 890771

*See [L]. Subsequently Jerison and Lee [JL] showed that there is a canonical way of choosing μ as well (by solving a CR variant of the Yamabe problem).

can be solved backwards in time over the interval $-r \leq t \leq 0$.

In other words Boutet's result says that the problem of extending f to a small neighborhood of X inside the complexification, $X_{\mathbb{C}}$, is equivalent to solving a diffusion problem in the wrong direction! Matt Stenzel and I showed in $[GS]_2$ that this result has some interesting connections with homogeneous Monge-Ampere. In this paper I will show that there is a form of Boutet's result which is true for an *arbitrary* real-analytic pseudoconvex domain; and this, too, will involve homogeneous Monge-Ampere in a fundamental way. The statement and proof of this result will be given in §5 and I will give my new proof of Lee's theorem in §4. As in $[GS]_1$ the main step in this proof will be the complexification of a solution of a certain *real* Monge-Ampere equation which I now want to describe: Let X and Y be real n-dimensional manifolds and consider the DeRham complex on $X \times Y$. By the Künneth theorem this complex is a double complex with an exterior derivative, d_x, that only involves the X-variables and an exterior derivative, d_y, that only involves the Y-variables. In particular, given a function, $\phi = \phi(x, y)$, on $X \times Y$ one gets a two-form, $d_x d_y \phi$, and, wedging this form with itself n times, a $2n$-form, $(d_x d_y \phi)^n$. Now let S be a hypersurface in $X \times Y$ and ϕ_0 a defining function for it. Suppose that ϕ_o satisfies:

$$(d_x d_y \phi_0)^{n-1} \wedge d_x \phi_0 \wedge d_y \phi_0 \neq 0 \tag{1.2}$$

on a neighborhood of S.* I will prove in §2 that, on every sufficiently small neighborhood of S, there exists a unique function, ϕ, such that $\phi - \phi_0$ vanishes to second order on S and

$$(d_x d_y \phi)^n = 0. \tag{1.3}$$

In other words given a surface, S, with the convexity property, (1.2), the Cauchy problem for (1.3), with initial data on S, can always be solved in a neighborhood of S. The proof will involve some ideas that have come up earlier in the work of Phong and Stein, [PS], and in my own work with Sternberg ([GS], Chapter 6) on Radon integral transforms; and I will explain what Monge-Ampere has to do with this subject in §2–3.

To conclude I would like to mention a number of recent articles on homogeneous Monge-Ampere dealing with issues that I've touched on here. These are, in addition to my two articles with Stenzel cited above, the article, [EM], of Epstein-Melrose and the articles, [LS] of Lempert-Szöke, [S] of Szöke and [Lem] of Lempert. In particular, in Lempert's article, it is shown that for the Monge-Ampere problem discussed in $[GS]_1$, $[GS]_2$ the analyticity assumptions are *necessary* as well as sufficient.

§2. Double fibrations.

*This condition depends only on S not on the choice of ϕ_0. It is the analogue in this "Künneth" theory of the Levi condition.

Let X and Y be n-dimensional manifolds and S a closed $(2n-1)$-dimensional submanifold of $X \times Y$. Let π and ρ be the restrictions to S of the projection maps of $X \times Y$ onto X and Y. The triple (S, π, ρ) is called a *double fibration* if both π and ρ are fiber mappings. I will assume that the conormal bundle of S is oriented and will denote by Γ the set of its positively-oriented vectors. Composing the inclusion, $\Gamma \longrightarrow T^*(X \times Y)$, with the projections of $T^*(X \times Y)$ and T^*X and T^*Y one gets maps

$$\pi_1 : \Gamma \longrightarrow T_0^*X \quad \text{and} \quad \rho_1 : \Gamma \longrightarrow T_0^*Y \tag{2.1}$$

of Γ onto the punctured cotangent bundles of X and Y.* The data, (S, π, ρ), are said to satisfy the *Bolker* condition if π_1 and ρ_1 are diffeomorphisms, in which case the composite mapping, $\rho_1 \circ \pi_1^{-1}$ is well-defined. Composing this mapping with the involution:

$$\sigma : T_0^*Y \longrightarrow T_0^*Y \quad , \quad \sigma(y, \eta) = (y, -\eta)$$

one gets a canonical transformation

$$\gamma : T_0^*X \longrightarrow T_0^*Y \tag{2.2}$$

which I will call the *canonical transformation associated with the double fibration* (S, π, ρ).

To check that the Bolker condition is satisfied, one has to check first that π_1 and ρ_1 are diffeomorphisms locally in the neighborhood of each point of Γ, and then check that they are one-one and onto. Often the second criterion is *implied* by the first. (This is so, for instance, if both X and Y are compact.) As for the first criterion, it is easy to see that if π_1 is locally a diffeomorphism at a point of Γ, ρ_1 is as well. This criterion can also be checked rather easily by the following means. Let $\phi = \phi(x, y)$ be a defining function of S i.e. let S be the subset of $X \times Y$ defined by the equation, $\phi(x, y) = 1$; and assume $d\phi_p \neq 0$ at all points, $p \in S$. Let $d_x d_y \phi$ be the two-form

$$\sum_{i,y=1}^{n} \frac{\partial^2 \phi}{\partial x_i \partial y_i} dx_i \wedge dy_j$$

Lemma. *For π_1 and ρ_1 to be local diffeomorphisms at all points of Γ it is necessary and sufficient that the $2n$-form*

$$(d_x d_y \phi)^{n-1} \wedge d_x \phi \wedge d_y \phi \tag{2.3}$$

be non-vanishing on a neighborhood of S.

I will leave the proof of this as an easy exercise. My goal in this section is to prove that if S satisfies the Bolker condition it has a defining function which satisfies, in addition to (2.3), the homogeneous Monge-Ampere equation described in the introduction:

*Given a manifold, M, we will denote by $T_0^*(M)$ the cotangent bundle of M with its zero section deleted.

Theorem 1. *Let $\mu : S \longrightarrow \Gamma$ be a section of Γ. Then there exists a unique defining function, ϕ, of S such that*

$$(d_x d_y \phi)^n \equiv 0 \tag{2.4}$$

on a neighborhood of S, and such that, in addition, $d\phi_p = \mu_p$ at all points, $p \in S$.*

Proof. Existence: There exists a unique homogeneous function of degree one on Γ which is identically equal to one on the image of μ. Lets denote this function by H_0. Under the diffeomorphism $T_0^*X \longrightarrow \Gamma$ this pulls back to a homogeneous function of degree one, H, on T_0^*X. Since (S, π, ρ) is a double fibration the fibers, $S_y = \rho^{-1}(y)$, above points of Y are $(n-1)$-dimensional submanifolds of X. Now, with y fixed, solve the Hamilton-Jacobi equation:

$$H(d\phi) = H\left(x, \frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}\right) = 1 \tag{2.5}$$

with the initial condition $\phi = 1$ on S_y.* This solution depends parametrically on y so it is really a function, $\phi = \phi(x, y)$, of *both* the x and the y variables and is well-defined in a neighborhood, U, of S. Let's show that it satisfies the Monge-Ampere equation and the required initial conditions. That it satisfies the initial conditions is equivalent to the assertion that $H_0(d\phi) = 1$ on S and this is equivalent to the assertion that, for y fixed, the equation $H(d\phi) = 1$ holds on X. To check that ϕ satisfies Monge-Ampere, we note that because H doesn't depend on y we can differentiate the identify

$$H\left(\frac{\partial \phi}{\partial x_1}, \ldots, \frac{\partial \phi}{\partial x_n}, x\right) = 1$$

with respect to y_i getting:

$$\sum_{j=1}^{n} \frac{\partial H}{\partial \xi_j}(d_x \phi, x) \frac{\partial^2 \phi}{\partial x_j \partial y_i} = 0$$

Since $\frac{\partial H}{\partial \xi}(x, \xi) \neq 0$ when $\xi \neq 0$ this implies that

$$\det\left(\frac{\partial^2 \phi}{\partial x_i \partial y_j}\right) = 0.$$

Uniqueness: Let ϕ be a defining function of S satisfying the given initial conditions. By assumption the map

$$\pi_1 : S \times \mathbb{R}^+ \longrightarrow T_0^*X$$

*In local coordinates this is just the Monge-Ampere equation $\det(\frac{\partial^2 \phi}{\partial x_i \partial y_j}) = 0$.

*Let's briefly review how this is done. The equation, $H = 1$, cuts out a hypersurface in the conormal bundle of S_y. This hypersurface is an isotropic submanifold of T^*X of dimension $n-1$, so if we take its flow-out with respect to the Hamiltonian flow, $\exp t\Xi_H$, we get an n-dimensional Lagrangian submanifold, Λ, of T^*X. In the vicinity of S_y Λ is the graph of an exact one form, $d\phi$, and if we normalize ϕ to be one on S_y this determines it uniquely.