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Geometric structures and characteristic forms

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On complex manifolds, we consider various holomorphic geometric structures such as affine structures, projective structures and conformal structures. When they admit such geometric structures, then their Chern classes satisfy certain formulae. For compact Kähler manifolds, these formulae were first found in the affine case by Atiyah [A], in the projective case by Gunning [G], and in the conformal case by Kobayashi-Ochiai [KO]. In the following, we shall introduce new characteristic forms defined by projective Weyl curvature tensors and conformal Weyl curvature tensors, and study their relations with the Chern forms. As byproducts, we obtain generalizations and refinements of the formulae quoted above. Details of this note will be found in [K1] and [K2].

Notation

$\Omega^p(E)$	:	the sheaf of germs of holomorphic p-forms with values in a holomorphic vector bundle E,
$\mathcal{O}(E) \simeq \Omega^0(E)$	:	the sheaf of germs of holomorphic sections of a holomorphic vector bundle E,
Θ	:	the sheaf of germs of holomorphic vector fields,
T	:	the sheaf of germs of differentiable vector fields,
$\mathcal{A}^r(G)$	:	the sheaf of germs of differentiable r-forms with values in a differentiable vector bundle G,
$\mathcal{A}^{p,q}(G)$	:	the sheaf of germs of differentiable (p,q)-forms with values in a differentiable vector bundle G.

1 Affine structures

Let X be a complex manifold of dimension $n \geq 1$. Take a locally finite open covering $\mathcal{U} = \{U_\alpha\}$ of X such that on each U_α, there is a system of local coordinates $z_\alpha = (z_\alpha^1, z_\alpha^2, \ldots, z_\alpha^n)$. Put

$$\varphi_{\alpha\beta} = z_\alpha \circ z_\beta^{-1}$$

and

$$\tau_{\alpha\beta} = \text{the Jacobian matrix of } \varphi_{\alpha\beta}.$$

On $U_\alpha \cap U_\beta$, we consider an $n \times n$-matrix-valued holomorphic 1-form

$$a_{\alpha\beta} = \tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta}.$$

It is well-known and easy to check that the set $\{a_{\alpha\beta}\}$ define an element of $H^1(X, \Omega^1(\mathrm{End}\Theta))$.

Definition 1.1 *The cohomology class*

$$a_X = \{a_{\alpha\beta}\} \in H^1(X, \Omega^1(\mathrm{End}\Theta))$$

is called the obstruction to the existence of holomorphic affine connections of X.

Definition 1.2 *For a complex manifold X with $a_X = 0$ there exists a (holomorphic) 0-cochain $\{a_\alpha\}$ such that $\delta\{a_\alpha\} = \{a_{\alpha\beta}\}$, which is called a holomorphic affine connection of X. If X has a holomorphic affine connection, we also say that X admits an affine structure. There always exists a C^∞ 0-cochain such that $\delta\{a_\alpha\} = \{a_{\alpha\beta}\}$ in the natural sense, where the a_α is an element of $\Gamma(U_\alpha, \mathcal{A}^{1,0}(\mathrm{End}T))$. The 0-cochain is called a C^∞ affine connection.*

Let $\theta = \{a_\alpha\}$ be a C^∞ affine connection. Then we have

$$a_{\alpha\beta} = a_\beta - \tau_{\alpha\beta}^{-1} a_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

The curvature form of the C^∞ affine connection

$$\Theta_\alpha = da_\alpha + a_\alpha \wedge a_\alpha$$

satisfies the equation

$$\Theta_\beta = \tau_{\alpha\beta}^{-1} \Theta_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta.$$

Let t be an indeterminate and A be an $n \times n$ matrix. Define polynomials φ_0, $\varphi_1, \ldots, \varphi_n$ by

$$\det(I - \frac{1}{2\pi i} tA) = \Sigma_{k=0}^n \varphi_k(A) t^k.$$

The Chern forms are defined by

$$c_q(\theta) = \varphi_q(\Theta_\alpha).$$

The following is a well-known fundamental fact.

Theorem 1.1 *For any $q = 0, 1, \ldots, n$, the Chern form $c_q(\theta)$ is a d-closed C^∞ $2q$-form. The corresponding de Rham cohomology class $[c_q(\theta)]$ is real and independent of the choice of the connection θ.*

Corollary 1.1 [A] *If a compact complex manifold with dimension $n \geq 1$ admits a holomorphic affine connection then all q-th Chern forms with $2q \geq n$ vanish. If , further, the manifold is of Kähler then all q-th Chern classes, $q \geq 1$, are zero.*

Proof. Note that the Chern forms defined by a holomorphic affine connection are holomorphic. Therefore if $2q > n$ then the q-th Chern form vanishes. Note that d-closed holomorphic n-form represents a real de Rham cohomology class only if it represents a zero class. Hence the n-th Chern class vanishes. If the manifold is of Kähler then any holomorphic form is harmonic. Since the Chern classes are real, they vanish by Hodge theory. ∎

2 Projective structures

In this section, we assume that $n = \dim X \geq 2$. We use the notation of section 1. On $U_\alpha \cap U_\beta$, we define a scalar-valued holomorphic 1-form

$$\sigma_{\alpha\beta} = (n+1)^{-1} d\log(\det \tau_{\alpha\beta}) = (n+1)^{-1}\mathrm{Trace}(\tau_{\alpha\beta}^{-1} d\tau_{\alpha\beta})$$

and an $n \times n$ matrix-valued holomorphic 1-form $\rho_{\alpha\beta}$ by

$$(\rho_{\alpha\beta})^j_k = \sigma_{\alpha\beta k} dz^j_\beta,$$

where

$$\sigma_{\alpha\beta} = \sigma_{\alpha\beta j} dz^j_\beta,$$

and $(A)^j_k$ indicates the (j,k)-component of the matrix A. Put

$$p_{\alpha\beta} = a_{\alpha\beta} - \rho_{\alpha\beta} - I \cdot \sigma_{\alpha\beta}, \tag{1}$$

where I is the identity matrix of size n. The 1-cochain $\{\sigma_{\alpha\beta}\}$ is a cocycle of $H^1(X, \Omega^1)$, and $\{\rho_{\alpha\beta}\}$ and $\{p_{\alpha\beta}\}$ are cocycles of $H^1(X, \Omega^1(\mathrm{End}\Theta))$.

Definition 2.1 *The cohomology class*

$$p_X = \{p_{\alpha\beta}\} \in H^1(X, \Omega^1(\mathrm{End}\Theta))$$

is called the obstruction to the existence of holomorphic projective connections of X.

Definition 2.2 *For a complex manifold X with $p_X = 0$ there exists a (holomorphic) 0-cochain $\{p_\alpha\}$ such that $\delta\{p_\alpha\} = \{p_{\alpha\beta}\}$, which is called a holomorphic projective connection of X. If X has a holomorphic projective connection, we also say that X admits a projective structure. There always exists a C^∞ 0-cochain $\{p_\alpha\}$ such that $\delta\{p_\alpha\} = \{p_{\alpha\beta}\}$ in a natural sense, where p_α is an element of $\Gamma(U_\alpha, \mathcal{A}^{1,0}(\mathrm{End}\Theta))$. The 0-cochain $\{p_\alpha\}$ is called a C^∞ projective connection.*

Let $\pi = \{p_\alpha\}$ be a C^∞ projective connection on X, that is, on each U_α there is an $n \times n$ matrix-valued C^∞ $(1,0)$-form p_α such that

$$p_\beta = p_{\alpha\beta} + \tau_{\alpha\beta}^{-1} p_\alpha \tau_{\alpha\beta} \quad \text{on} \quad U_\alpha \cap U_\beta. \tag{2}$$

We write the (j,k)-component of p_α and $p_{\alpha\beta}$ as

$$\begin{aligned} (p_\alpha)^j_k &= p^j_{\alpha ik} dz^i_\alpha, \\ (p_{\alpha\beta})^j_k &= p^j_{\alpha\beta ik} dz^i_\beta. \end{aligned}$$

By the definition (1), we have $p^j_{\alpha\beta kl} = p^j_{\alpha\beta lk}$. Therefore it is easy to see that the $n \times n$ matrix-valued 1-form q_α defined by

$$(q_\alpha)^j_k = p^j_{\alpha ki} dz^i_\alpha$$

is also a projective connection. Hence $\{2^{-1}(p_\alpha + q_\alpha)\}$ is also a projective connection. Therefore we may assume that

$$p^j_{\alpha kl} = p^j_{\alpha lk} \tag{3}$$

holds. Since $\mathrm{Trace}(p_\beta) = 0$, it follows from (2) that

$$\mathrm{Trace}(p_\beta) = \mathrm{Trace}(p_\alpha).$$

Since $\{p_\alpha - n^{-1}\mathrm{Trace}(p_\alpha) I\}$ is also a projective connection, we may assume that

$$p^j_{\alpha ij} = 0 \tag{4}$$

holds. The projective connection satisfying (3) is said to be *normal.* The projective connection satisfying (4) is said to be *reduced.* Thus any complex manifold admits a normal reduced C^∞ projective connection. As we see easily from the above argument, *if a complex manifold admits a holomorphic projective connection, the manifold admits also a normal reduced holomorphic projective connection.* In the following in this section, we consider only normal reduced (holomorphic or C^∞) projective connections.

The projective Weyl curvature tensor $\{W_\alpha\}$ associated with a (normal reduced) C^∞ projective connection $\pi = \{p_\alpha\}$ is defined by