CHRISTINE LAURENT-THIÉBAUT
JÜRGEN LEITERER

Uniform estimates for the Cauchy-Riemann equation on q-concave wedges

Astérisque, tome 217 (1993), p. 151-182
<http://www.numdam.org/item?id=AST_1993__217__151_0>

UNIFORM ESTIMATES FOR THE CAUCHY-RIEMANN EQUATION ON q-CONCAVE WEDGES

Christine LAURENT-THIÉBAUT and Jürgen LEITERER

0. Introduction

This article is the continuation of [L-T/Le]. Both papers are preliminary works for a systematic study of the tangential Cauchy-Riemann equation on real submanifolds from the viewpoint of uniform estimates and by means of integral formulas. For this study we have to solve the Cauchy-Riemann equation with uniform estimates on q-convex and q-concave wedges in $\mathbb{C}^n$ (for historical remarks, see the introduction to [L-T/Le]). Whereas [L-T/Le] is devoted to q-convex wedges, here we study q-concave wedges.

The main result of the present paper can be formulated as follows. Let $G \subseteq \mathbb{C}^n$ be a domain, q an integer with $1 \leqslant q \leqslant n-1$, and $\varphi_1, \dots, \varphi_N$ a collection of real C^2 functions on G satisfying the following three conditions :

(i) $E := \{ z \in G : \varphi_1(z) = \cdots = \varphi_N(z) = 0 \} \neq \emptyset$;

(ii) $d\varphi_1(z) \wedge \cdots \wedge d\varphi_N(z) \neq 0$ for all $z \in G$;

(iii) If $\lambda = (\lambda_1, \dots, \lambda_N)$ is a collection of non-negative real numbers with $\lambda_1 + \cdots + \lambda_N = 1$, then, at all points in G, the Levi form of the function

$$\lambda_1\varphi_1 + \cdots + \lambda_N\varphi_N$$

has at least $q+1$ positive eigenvalues.

Set

$$D = \bigcap_{j=1}^{N} \{ z \in G : \varphi_j(z) > 0 \} \tag{0.1}$$

and

$$\Omega = \bigcup_{j=1}^{N} \{ z \in G : \varphi_j(z) > 0 \} . \tag{0.2}$$

Further, for $\xi \in \mathbb{C}^n$ and $R > 0$, we denote by $B_R(\xi)$ the open ball of radius R in $\mathbb{C}^n$ centered at ξ. Then Theorems 5.6, 5.7 and 6.6 of the present work imply the following

0.1. THEOREM. — *For each point $\xi \in E$ there exists a radius $R > 0$ such that :*

(a) *If $q-N \geqslant 0$, then each holomorphic function on D extends holomorphically to $D \cup B_R(\xi)$;*

(b) *If $q-N \geqslant 1$ and f is a continuous $\overline{\partial}$-closed (n,r)-form with $1 \leqslant r \leqslant q-N$ on D, then there exists a continuous $(n,r-1)$-form u on $D \cap B_R(\xi)$ with*

$$\overline{\partial} u = f \quad \text{on} \quad D \cap B_R(\xi) . \tag{0.3}$$

Moreover if, for some β with $0 \leqslant \beta < 1$, f satisfies the estimate

$$\| f(\zeta) \| \leqslant [\operatorname{dist}(\zeta, \partial D)]^{-\beta}, \quad \zeta \in D, \tag{0.4}$$

then the solution u of (0.3) can be given by an explicit integral operator and, for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ (independent of f) such that :

If $0 \leqslant \beta < 1/2$, then u is Hölder continuous with exponent $1/2-\beta-\varepsilon$ on $\overline{D \cap B_R(\xi)}$ and

$$\|u\|_{1/2-\beta-\varepsilon, \overline{D \cap B_R(\xi)}} \leqslant C_\varepsilon \sup_{\zeta \in D} \| f(\zeta) \| [\operatorname{dist}(\zeta, \partial D)]^{\beta}, \tag{0.5}$$

where $\| \cdot \|_{1/2-\beta-\varepsilon, \overline{D \cap B_R(\xi)}}$ is the Hölder norm with exponent $1/2-\beta-\varepsilon$ on $\overline{D \cap B_R(\xi)}$.

If $1/2 \leqslant \beta < 1$, then

$$\sup_{z \in D} \| u(z) \| [\operatorname{dist}(z, \partial D)]^{\beta - 1/2 + \varepsilon} \leqslant C_\varepsilon \sup_{\zeta \in D} \| f(\zeta) \| [\operatorname{dist}(\zeta, \partial D)]^{\beta} . \tag{0.6}$$

Note that the radius R and the constant C_ε in Theorem 0.1 depend continuously on $\varphi_1, \dots, \varphi_N$ with respect to the C^2 topology.

Theorem 0.1 implies the following corollary for the domain Ω defined by (0.2) :

0.2. COROLLARY. — *For each point $\xi \in E$ there exists a radius $R > 0$ such that :*

(i) *If $q \geqslant 1$, then each holomorphic function on Ω extends holomorphically to $\Omega \cup B_R(\xi)$;*

(ii) *If $q \geqslant 2$ and f is a continuous $\overline{\partial}$-closed (n,r)-form with $1 \leqslant r \leqslant q-1$ on Ω, then there is a continuous $(n,r-1)$-form u on $\Omega \cap B_r(\xi)$ with*

$$\overline{\partial} u = f \quad \text{on} \quad \Omega \cap B_r(\xi) . \tag{0.7}$$

It is easy to see that, for $r = 1$, estimates (0.5) and (0.6) (with Ω instead of D) hold also in this corollary. We do not know whether this is true for $r \geqslant 2$.

For the smooth case ($N = 1$) Theorem 0.1 was obtained by Lieb [Li]. We prove Theorem 0.1 by means of integral formulas which are obtained combining the construction of Lieb [Li] with the construction of Range and Siu [R/S]. The main problem then consists in the proof of the estimates. Fortunately, in large parts, this proof is parallel to the corresponding proof in the q-convex case which is carried out in [L-T/Le]. Note that, in both proofs, an idea of Henkin plays a very important role (see the introduction to [L-T/Le]). Note also that in the survey article [He] of Henkin a global result, corresponding to the important special case $\beta = 0$, $\varepsilon = \frac{1}{2}$ of Theorem 0.1 is formulated (see [He] th. 8-12 d)).

Finally we want to compare our results with the work [G] of Grauert. He studied domains of type Ω defined by (0.2), where instead of condition *(iii)* the following stronger hypothesis is used :

(iii)' There is a fixed $(q+1)$-dimensional subspace T of $\mathbb{C}^n$ such that, for all $j = 1, \dots, N$ and $z \in G$, the Levi form φ_j is positive definite on T.

Under this hypothesis, Corollary 0.2 follows from Satz 1 in [G]. Note that the conclusion of Satz 1 in [G] is essentially stronger than the conclusion of our Corollary 0.2 : we can solve $\overline{\partial} u = f$ only on the smaller set $\Omega \cap B_r(\xi)$ if f is given on Ω, whereas Grauert proves the existence of a basis of Stein neighborhoods U of ξ such that, if f is given on $\Omega \cap U$, the equation $\overline{\partial} u = f$ can be solved on the same set $\Omega \cap U$. In the smooth case ($N = 1$) such a solution without shrinking of the domain is possible also with estimates as in Theorem 0.1 (see Theorem 14.1 in [He/Le 2]). On the other hand, it is not clear whether one can solve (even without estimates) the $\overline{\partial}$-equation without shrinking of the domain in the situation of Theorem 0.1 if $N \geqslant 2$. Note also that the statement of Theorem 0.1 under the stronger condition *(iii)'* and without estimates and with shrinking of the domain can be obtained also from Satz 1 in [G].

1. Preliminaries

1.1. — For $z \in \mathbb{C}^n$ we denote by $z_1, \dots, z_n$ the canonical complex coordinates of z. We write $\langle z, w \rangle = z_1 w_1 + \cdots + z_n w_n$ and $|z| = \langle z, z \rangle^{1/2}$ for $z, w \in \mathbb{C}^n$.

1.2. — Let M be a closed real C^1 submanifold of a domain $\Omega \subseteq \mathbb{C}^n$, and let $\zeta \in M$. Then we denote by $T_\zeta^{\mathbb{C}}(M)$ the *complex*, and by $T_\zeta^{\mathbb{R}}(M)$ the *real* tangent space of M at ζ. We identify these spaces with subspaces of $\mathbb{C}^n$ as follows : if $\rho_1, \dots, \rho_N$ are real C^1 functions in a neighborhood U_ζ of ζ such that $M \cap U = \{ \rho_1 = \cdots = \rho_N = 0 \}$ and

$d\rho_1(\zeta) \wedge \cdots \wedge d\rho_N(\zeta) \neq 0$, then

$$T_\zeta^{\mathbf{C}}(M) = \left\{ t \in \mathbf{C}^n : \sum_{\nu=1}^{n} \frac{\partial \rho_j(\zeta)}{\partial \zeta_\nu} t_\nu = 0 \ \text{ for } \ j = 1, \dots, n \right\}$$

and

$$T_\zeta^{\mathbf{R}}(M) = \left\{ t \in \mathbf{C}^n : \sum_{\nu=1}^{2n} \frac{\partial \rho_j(\zeta)}{\partial x_\nu} x_\nu(t) = 0 \ \text{ for } \ j = 1, \dots, n \right\},$$

where $x_1, \dots, x_{2n}$ are the real coordinates on $\mathbf{C}^n$ with $t_\nu = x_\nu(t) + i x_{\nu+n}(t)$ for $t \in \mathbf{C}^n$ and $\nu = 1, \dots, n$.

1.3. — Let $\Omega \subseteq \mathbf{C}^n$ be a domain and ρ a real C^2 function on Ω. Then we denote by $L_\rho(\zeta)$ the Levi form of ρ at $\zeta \in \Omega$, and by $F_\rho(\cdot, \zeta)$ the Levi polynomial of ρ at $\zeta \in \Omega$, i.e.

$$L\rho(\zeta)t = \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \overline{\zeta}_j \partial \zeta_k} \overline{t}_j t_k$$

$\zeta \in \Omega,\ t \in \mathbf{C}^n$, and

$$F_\rho(z, \zeta) = 2 \sum_{j=1}^{n} \frac{\partial \rho(\zeta)}{\partial \zeta_j} (\zeta_j - z_j) - \sum_{j,k=1}^{n} \frac{\partial^2 \rho(\zeta)}{\partial \zeta_j \partial \zeta_k} (\zeta_j - z_j)(\zeta_k - z_k)$$

$\zeta \in \Omega,\ z \in \mathbf{C}^n$. Recall that by Taylor's theorem (see, e.g., Lemma 1.4.13 in [He/Le 1])

$$\operatorname{Re} F_\rho(z, \zeta) = \rho(\zeta) - \rho(z) + L_\rho(\zeta)(\zeta - z) + o(|\zeta - z|^2)\ . \tag{1.1}$$

1.4. — Let $J = (j_1, \dots, j_\ell)$, $1 \leqslant \ell < \infty$, be an ordered collection of elements in $\mathbf{N} \cup \{*\}$. Then we write $|J| = \ell$, $J(\hat{\nu}) = (j_1, \dots, j_{\nu-1}, j_{\nu+1}, \dots, j_\ell)$ for $\nu = 1, \dots, \ell$, and $j \in J$ if $j \in \{ j_1, \dots, j_\ell \}$.

1.5. — Let $N \geqslant 1$ be an integer. Then we denote by $P(N)$ the set of all ordered collections $K = (k_1, \dots, k_\ell)$, $\ell \geqslant 1$, of integers with $1 \leqslant k_1, \dots, k_\ell \leqslant N$, and by $P(N, *)$ the set of all ordered collections $K = (k_1, \dots, k_\ell)$, $\ell \geqslant 1$ such that either $K \in P(N)$ or for a $\nu \in \{ 1, \dots, \ell \}$, $k_\nu = *$ and $K(\hat{\nu}) \in P(N)$ as well as $K = (*)$. We call $P'(N)$ the subset of all $K = (k_1, \dots, k_\ell) \in P(N)$ with $k_1 < \cdots < k_\ell$ and $P'(N, *)$ the subset of all $K = (k_1, \dots, k_\ell)$ where either $K \in P'(N)$ or $1 \leqslant k_1 < \cdots < k_{\ell-1} \leqslant N$ and $k_\ell = *$, i.e. $K_{(\hat{\ell})} \in P'(N)$ and $K = K_{(\hat{\ell})}*$, as well as $K = (*)$.

1.6. — Let $J = (j_1, \dots, j_\ell)$, $1 \leqslant \ell < \infty$, be an ordered collection of integers with $0 \leqslant j_1 < \cdots < j_\ell$. Then we denote by Δ_J (or $\Delta_{j_1 \cdots j_\ell}$) the simplex of all sequences $\{ \lambda_j \}_{j=0}^{\infty}$ of numbers $0 \leqslant \lambda_j \leqslant 1$ such that $\lambda_j = 0$ if $j \notin J$ and $\Sigma \lambda_j = 1$. We orient Δ_J by the form $d\lambda_{j_2} \wedge \cdots \wedge d\lambda_{j_\ell}$ if $\ell \geqslant 2$, and by +1 if $\ell = 1$.

Further Δ_{J*} (or $\Delta_{j_1 \cdots j_\ell *}$) will be the simplex of all sequences $\{ \lambda_j \}_{j=0}^{\infty} \cup \{ \lambda_* \}$ of numbers $0 \leqslant \lambda_j \leqslant 1$, $0 \leqslant \lambda_* \leqslant 1$ such that $\lambda_j = 0$ if $j \notin J$ and $\sum_{j=0}^{\infty} \lambda_j + \lambda_* = 1$. We orient Δ_{J*} by the form $d\lambda_{j_2} \wedge \cdots \wedge d\lambda_{j_\ell} \wedge d\lambda_*$.