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ON THE ENVELOPES OF HOLOMORPHY OF STRICTLY LEVI-CONVEX HYPERSURFACES

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Introduction

We shall be concerned with the subject of holomorphic continuation of CR-functions from a relatively open part of the boundary of a strongly pseudoconvex domain.

Let M be a Stein manifold of dimension $n \geq 2$, $D \subset\subset M$ a $\mathcal{C}^2$-bounded strongly pseudoconvex domain and K a proper closed subset of the boundary bD of D.

It is well-known that, due to the strict Levi-convexity of $bD \setminus K$, there exists an open set $U \subset D$, having $bD \setminus K$ as a part of its boundary, such that every continuous CR-function on $bD \setminus K$ has a unique continuous extension to $(bD \setminus K) \cup U$ which is holomorphic on U. The existence of U is referred to as the H. Lewy's extension phenomenon.

More recents results yield sharper information on U; in particular it has been shown that the open set $D \setminus \widehat{K}_{\overline{D}}$ ($\widehat{K}_{\overline{D}} = \mathcal{O}(\overline{D})$-hull of K) is such a U with the mentioned features (see [11, 6] and the references therein).

For $n = 2$ it is also known that $D \setminus \widehat{K}_{\overline{D}}$ has another independent property: it is pseudoconvex (see [8, 9, 10]). This, combined with the above, implies at once the following noteworthy result:

(I) *For $n = 2$ the envelope of holomorphy of $bD \setminus K$ is $\overline{D} \setminus \widehat{K}_{\overline{D}}$.*

Remark. Here above and throughout the continuation we speak of envelopes of holomorphy of non-open subsets of M. We recall that in general the envelope of holomorphy $E(S)$ of an arbitrary subset S of a Stein manifold can be given a precise definition as the union of the components of $\widetilde{S} = spec(\mathcal{O}(S))$ which meet S (see [5]). However, in the case of our concern where $S = bD \setminus K$, for the purposes of this paper the envelope of holomorphy may be simply understood as the disjoint union of $bD \setminus K$ and the envelope of holomorphy $E(U)$ of an open set U as specified above, regarded as a holomorphic extension of U.

An immediate consequence of (I) is:

$(I)'$ For $n = 2$, in order that K be removable, in the sense that each continuous CR-function f on $bD \setminus K$ may have a continuous extension $F \in \mathcal{C}^0(\overline{D} \setminus K) \cap \mathcal{O}(D)$, it is necessary and sufficient that $\widehat{K}_{\overline{D}} = K$, i.e. that K be $\mathcal{O}(\overline{D})$-convex.

On the other hand, for $n \geq 3$ it is not true in general that $D \setminus \widehat{K}_{\overline{D}}$ is pseudoconvex, as simple examples show, and hence the extension of (I) to general $n \geq 2$ fails to be valid. Indeed Corollary 2 below specifies the necessary and sufficient conditions for $D \setminus \widehat{K}_{\overline{D}}$ to be pseudoconvex when $n \geq 3$. Also the extension of $(I)'$ to general $n \geq 2$ does not hold, since for $n \geq 3$ $\mathcal{O}(\overline{D})$-convexity is no longer necessary for removability: for example every Stein compactum on bD is removable for $n \geq 3$ (see [11]).

In fact, when $n \geq 3$ no theorem of the kind of (I), to the effect of describing the envelope of holomorphy of $bD \setminus K$ for an arbitrary compact set $K \subset bD$, is known, and it is even unknown, as far as we can say, whether it is always true that $bD \setminus K$ should have a single-sheeted envelope of holomorphy.[1]

As regards $(I)'$, on the contrary, an extension to $n \geq 2$ has been recently established (see [7]). It can be stated as follows:

(II) For $n \geq 2$, in order that K be removable it is necessary and sufficient that $H^{n-1}(K; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(\overline{D}; \mathcal{O}) \to H^{n-2}(K; \mathcal{O})$ have dense image.

Since for $n = 2$ the vanishing of $H^1(K; \mathcal{O})$ is equivalent to the condition that K be holomorphically convex (see [5]), it follows that (II) is indeed an extension of $(I)'$ to general $n \geq 2$. Note that, since $\overline{D}$ is a Stein compactum, and hence $H^q(\overline{D}; \mathcal{O}) = 0$ for $q \geq 1$, when $n \geq 3$ the condition on the restriction map amounts to having ${}^{\sigma}H^{n-2}(K; \mathcal{O}) = 0$, where the suffix σ means the associated separated space.

[1] *Added July 19, 1993.* Recently E.M. Chirka and E.L. Stout [Removable Singularities in the Boundary (to appear)] gave an example of a $\mathcal{C}^\infty$-bounded strongly pseudoconvex domain $D \subset\subset \mathbb{C}^{2m}$, $m \geq 2$, and a compact set $K \subset bD$, with $bD \setminus K$ being connected, such that the envelope of holomorphy of $bD \setminus K$ is not single-sheeted.

(II) gives a first answer to the question of finding, for general $n \geq 2$, the envelope of holomorphy of $bD \setminus K$. In fact it states a necessary and sufficient condition on K in order that the envelope may be the whole $\overline{D} \setminus K$. Here we shall establish a sharper result of this kind, which includes both (I) and (II) as particular cases, namely we shall prove the following theorem.

Theorem. *Let $n \geq 2$ and let E be a compact set such that $K \subset E \subset \widehat{K}_{\overline{D}}$. Then, in order that $\overline{D} \setminus E$ may be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that the following conditions should be satisfied:*

(1) *The restriction map $H^q(E; \mathcal{O}) \to H^q(K; \mathcal{O})$ is bijective for $q \leq n-3$ and is injective with closed image for $q = n-2$.*

(2) *$H^{n-1}(E; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(\overline{D}; \mathcal{O}) \to H^{n-2}(E; \mathcal{O})$ has dense image.*

It is plain that this theorem implies (II): just take in it $E = K$. On the other hand, for $n = 2$ Condition (2) means that $E = \widehat{K}_{\overline{D}}$, and then Condition (1) amounts to saying that the restriction map $\mathcal{O}(\widehat{K}_{\overline{D}}) \to \mathcal{O}(K)$ should be injective with closed image, which indeed can be shown to be automatically true (see [8]); therefore for $n = 2$ the theorem does reduce to (I).

We wish to mention a couple of straightforward further consequences of the theorem. If we apply it to the case that $n \geq 3$ and E is holomorphically convex (*e.g.* a Stein compactum), on account of the vanishing of $H^q(E; \mathcal{O})$ for $q \geq 1$, we get at once:

Corollary 1. *Let $n \geq 3$ and let E be a holomorphically convex compact set such that $K \subset E \subset \widehat{K}_{\overline{D}}$. Then, in order that $\overline{D} \setminus E$ be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$, that $H^{n-2}(K; \mathcal{O})$ be separated and that E be the envelope of holomorphy of K.*

In particular we can state:

Corollary 2. *For $n \geq 3$, in order that $\overline{D} \setminus \widehat{K}_{\overline{D}}$ be the envelope of holomorphy of $bD \setminus K$, it is necessary and sufficient that $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$, that $H^{n-2}(K; \mathcal{O})$ be separated and that $\widehat{K}_{\overline{D}}$ be the envelope of holomorphy of K.*

Remarks. (*i*) The cohomological conditions on K in the preceding corollaries can be shown to be equivalent to the following:

$$H^{n-2}(M \setminus K; \mathcal{O}) \text{ is separated, if } n = 3;$$
$$H^{q}(M \setminus K; \mathcal{O}) = 0 \text{ for } 2 \leq q \leq n-2, \text{ if } n \geq 4.$$

Moreover we recall that $H^2(M \setminus K; \mathcal{O})$ is separated if and only if $\bar{\partial}\mathcal{E}^{0,1}(M \setminus K)$ is a closed subspace of $\mathcal{E}^{0,2}(M \setminus K)$.

(*ii*) It is not possible to omit, in the preceding corollaries, the requirement that $H^{n-2}(K; \mathcal{O})$ should be separated. As a matter of fact, consider the open unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$, $n \geq 3$, and the compact sets $K = b\mathbb{B}_n \cap \{z \in \mathbb{C}^n : \mathcal{I}m(z_{n-1}) = 0, z_n = 0\}$, $E = \overline{\mathbb{B}}_n \cap \{z \in \mathbb{C}^n : \mathcal{I}m(z_{n-1}) = 0, z_n = 0\}$. It is readily seen that K is removable, and hence the envelope of holomorpy of $b\mathbb{B}_n \setminus K$ is not $\overline{\mathbb{B}}_n \setminus E$, but the whole $\overline{\mathbb{B}}_n \setminus K$. On the other hand E is both the envelope of holomorphy and the polynomial hull of K, moreover one has $H^q(K; \mathcal{O}) = 0$ for $1 \leq q \leq n-3$ and ${}^{\sigma}H^{n-2}(K; \mathcal{O}) = 0$. Indeed the point is that in this case $H^{n-2}(K; \mathcal{O})$ is not separated.

1. Preliminaries

Before going into the proof of the theorem we need some preliminary results. We shall use the notation that, given a compact set $E \subset M$, $\Phi(E)$, or simply Φ when no confusion can arise, denotes the paracompactifying family of supports in $M \setminus E$ of all the relatively closed subsets of $M \setminus E$ whose closure in M is compact, that is $\Phi = c \cap (M \setminus E)$, where c denotes the family of compact subsets of M.

Lemma 1. *For $n \geq 2$, if $M \setminus E$ is connected, the following facts are equivalent:*

(*a*) *$H^{n-1}(E; \mathcal{O}) = 0$ and the restriction map $H^{n-2}(M; \mathcal{O}) \to H^{n-2}(E; \mathcal{O})$ has dense image.*

(*b*) $H^1_{\Phi}(M \setminus E; \mathcal{O}) = 0$.

We have already established this result in [7], where it is needed for the proof of (*II*), so we refer to [7] for its proof.

Lemma 2. *For $n \geq 2$, if D, $E \subset M$ are a pseudoconvex domain and a compact set, respectively, the following facts are equivalent:*

(*α*) *The restriction map $H^q(\overline{D} \cap E; \mathcal{O}) \to H^q(bD \cap E; \mathcal{O})$ is bijective for $q \leq n-3$ and injective for $q = n-2$, moreover the space $H^{n-1}_c(D \cap E; \mathcal{O})$ is separated;*