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Straightening of Arcs

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dédié à Horace Bénédict de Saussure

pour son oeuvre sur l' hibernation des marmottes.

Introduction.

Any smooth arc γ in $\mathbb{C}^n$ is polynomially convex ([4], or [5]). And one can approximate any continuous function on γ by polynomials. Our goal is to show that if $n \geq 2$, under a global biholomorphic change of variables, an arc can always be "straightened" (approximately mapped to a line segment). This makes polynomial convexity and polynomial approximation trivial, unfortunately we need to use polynomial convexity in our proof. Here is a precise statement.

Proposition *Let* Γ *be a smooth* $(\mathcal{C}^\infty)$ *arc in* $\mathbb{C}^n (n \geq 2)$, *parametrized by* $\gamma : [0,1] \to \mathbb{C}^n$. *There exists* (T_j) *a sequence of automorphisms of* $\mathbb{C}^n$ *so that:* $T_j \circ \gamma$ *converges, in* $\mathcal{C}^\infty$ *topology, to the map* $t \mapsto (t, 0, \ldots, 0)$, *and the restriction of* $\overset{-1}{T}_j$ *to* $[0,1] \times \{0, \ldots, 0\})$ *(identified with* $[0,1]$*) converges to* γ *in* $\mathcal{C}^\infty$ *topology.*

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Remarks.

1) The Proposition above can be generalized to smooth totally real disks (of real dimension $k, k \leq n$) which are polynomially convex. In [2], a joint work with F. Forstnerič, the case of totally real manifolds is studied in much greater generality (in the real analytic setting).

2) There is a reason why we want not only the convergence of T_j on Γ, but also the convergence of $\overset{-1}{T}_j$ on $[0,1] \times \{(0,\dots,0)\}$. This is explained in II.

I. Proof of the Proposition

In I.1 we consider the case of real analytic arcs. For real analytic arcs, the Proposition follows very easily from a recent theorem by Andersen and Lempert [1], (see also [2]). In I.2 the case of smooth arcs is considered.

The theorem by Andersen and Lempert is this following:

Theorem (Andersen - Lempert) *Let T be a biholomorphic map from a star shaped domain Ω onto a Runge domain Ω' (in $\mathbb{C}^n, n \geq 2$). Then T can be uniformly approximated on compact sets in Ω by (global) automorphisms of $\mathbb{C}^n$.*

I.1 The real analytic case

We suppose that γ is a real analytic map from $[0,1\}$ into $\mathbb{C}^n, \dot{\gamma} \neq 0$ and γ is $1-1$. Set $J = [0,1] \times \{(0,\dots,0)\} \subset \mathbb{C}^n$. We identify $[0,1]$ with J. Then we can extend γ to a holomorphic map $\tilde{\gamma}$ defined in some neighborhood U of J in $\mathbb{C}^n$, and $1-1$. For example, we can set

$$\tilde{\gamma}(z_1,\dots,z_n) = \gamma(z_1) + \sum_{j=2}^{n} z_j \alpha_j(z_1).$$

In this formula γ denotes the holomorphic extension of γ to a neighborhood of $[0,1]$ in $\mathbb{C}$, and one has to take the ($\mathbb{C}^n$ valued) maps α_j holomorphic and so that for any $t \in [0,1]$ the vectors $\left(\dot\gamma(t), \alpha_2(t), \ldots, \alpha_n(t)\right)$ are linearly independent.

Let φ be a function defined on $\mathbb{R}$, which is 0 in $[0,1]$ and strictly positive and convex off $[0,1]$. Set $\rho(z_1, \ldots, z_n) = \varphi(x_1) + |y_1|^2 + \sum_{j=2}^{n} |z_j|^2$. For $\epsilon > 0$ let $U_\epsilon = \{z \in \mathbb{C}^n,\ \rho(z) < \epsilon\}$. This is a convex and strictly pseudoconvex neighborhood of J. For ϵ smaller than some ϵ_0 positive, $U_\epsilon \subset U$. We claim that for ϵ small enough $\tilde\gamma(U_\epsilon)$ is Runge. Indeed, since Γ is polynomially convex, Γ has a basis of Stein neighborhood which are Runge. Fix such a neighborhood $V \subset \tilde\gamma(U)$. Take ϵ so small that $\tilde\gamma(U_\epsilon) \subset V$. Then $\tilde\gamma(U_\epsilon)$ is Runge in V since it is defined by the inequality $\rho \circ \tilde\gamma^{-1} < \epsilon$ and $\rho \circ \tilde\gamma^{-1}$ is strictly plurisubharmonic ([3] Theorem 4.3.2).

Since V is Runge in $\mathbb{C}^n, \tilde\gamma(U_\epsilon)$ is Runge in $\mathbb{C}^n$, as desired.

We can apply the Andersen Lempert theorem to approximate the restriction of $\tilde\gamma$ to U_ϵ, uniformly on compact sets, by a sequence (S_j) of biholomorphisms of $\mathbb{C}^n$. Finally, set $T_j = \overset{-1}{S}_j$. We have better than convergence in the $\mathcal{C}^\infty$ topology on Γ or J. We have uniform convergence on neighborhoods of J and Γ, $\overset{-1}{T}_j$ converges to $\tilde\gamma$ on U_ϵ and (by the implicit function theorem) T_j converges, uniformly on compact sets to $\overset{-1}{\tilde\gamma}$ on $\tilde\gamma(U_\epsilon)$. Hence $T_j \circ \gamma$ converges to the map $t \mapsto (t, 0, \ldots, 0)$.

II.2 The smooth case

A very natural idea is to approximate smooth arcs by real analytic arcs, and then to straighten the real analytic arcs following I.1. This is what we are going to do, but it needs to be done with care (I did not find a trivial trick). In particular, one cannot expect uniform convergence on neighborhoods of Γ and $[0,1]$ as in the real analytic case. We first prove a Lemma to handle this question of convergence of maps defined on shrinking neighborhoods.

II.2.1 Lemma: *Let $K \subset \Omega \subset \mathbb{R}^p$ K convex and compact, Ω open. Set $\Omega_j = \{z \in \Omega, \operatorname{dist}(z,K) < \frac{1}{j}\}$. Let χ be a diffeomorphism from Ω into $\mathbb{R}^p$. Set $K' = \chi(K)$. Let χ_j be a sequence of smooth maps $\chi_j : \Omega_j \longrightarrow \mathbb{R}^p$ so that $\|\chi_j - \chi\|_{C^1(\Omega_j)} \leq \frac{C}{j^2}$. Then, for j large enough, $K' \subset \chi_j(\Omega_j)$, and χ_j is a diffeomorphism. And if $\|\chi_j - \chi\|_{C^k(\Omega_j)}$ tends to 0 as j tends to ∞, so does $\|\overset{-1}{\chi}_j - \overset{-1}{\chi}\|_{C^k(K')}$.*

(By $\|\Psi\|_{C^k(K')}$, we mean $\sup\limits_{\substack{|\alpha|\leq k\\ x\in K'}} |D^\alpha \Psi(x)|$, i.e. the $\mathcal{C}^k$ norm on jets which is stronger than the $\mathcal{C}^k$ norm of the restrictions)

Proof: Shrinking Ω if needed, we can assume that Ω is convex and that there is a constant $A > 0$ so that for all x and $y \subset \Omega$

$$\frac{1}{A}|x-y| \leq |\chi(x) - \chi(y)| \leq A\,|x-y|\,.$$

For j large enough (so that $\frac{C'}{j^2} < \frac{1}{2A}$) one gets

$$\frac{1}{2A}|x-y| \leq |\chi_j(x) - \chi_j(y)| \leq 2A\,|x-y|\,.$$

Hence χ_j is a diffeomorphism (whose Jacobian together with its inverse is bounded). Let $a \in K$, $S_j = \{z \in \mathbb{C}^n \ \ |z-a| = \frac{1}{j}\}$. The image of S_j under χ is a topological sphere which "contains" the ball of radius $\frac{1}{2Aj}$ centered at $\chi(a)$. For any $z \in S_j$, $|\chi(z) - \chi_j(z)| \leq \frac{C}{j^2}$.