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Introduction

Zariski's problem on cancellation (by an affine space) is usually formulated as follows * :

Let $\mathbf{X} \times \mathbf{A}^n \simeq \mathbf{Y} \times \mathbf{A}^n$ *be an isomorphism of algebraic varieties. Does it follows that* $\mathbf{X} \simeq \mathbf{Y}$ *?*

In general, the answer is negative even for surfaces over $\mathbf{C}$ **[Da]** , **[tDi]** . In an important special case, when $\mathbf{Y} = \mathbf{A}^k$, it is known only that the answer is positive for $\mathbf{k} \leq \mathbf{2}$ (M. Miyanishi - T. Sugie and T. Fujita, see **[Fu 2]** or **[Km]**).

It was C. P. Ramanujam, who in his earlier attempt to prove the latter result noticed a connection of the problem with the question of existence of exotic algebraic structures on affine spaces **[Ra]** . The main theorem in **[Ra]** on a characterization of the affine plane implies that the only complex algebraic structure on $\mathbf{R}^4$ is the standard structure of $\mathbf{C}^2$. (The proof of this theorem contains a great deal of tools that are used now in a study of acyclic algebraic surfaces.) Producing the first example of a topologically contractible smooth complex algebraic surface $\mathbf{X}$, non-isomorphic to $\mathbf{C}^2$, C. P. Ramanujam remarked that by the h-cobordism theorem the threefold $\mathbf{X} \times \mathbf{C}$ is diffeomorphic to $\mathbf{C}^3$, but it is not isomorphic as algebraic variety to $\mathbf{C}^3$ provided that the above version of the cancellation problem is answered affirmatively. Thus, this does lead to an exotic complex algebraic structure on $\mathbf{R}^6$.

In 1987–1989 many new examples of acyclic and contractible algebraic surfaces were constructed (see for instance, **[Gu Mi]**, **[tDi Pe]**, **[Su]**, **[Za 2]**). In the **Appendix** to this paper we shall describe two countable series of examples in which each surface $\mathbf{X}$ carries a family of curves $\mathbf{X} \to \mathbf{C}$ with a generic fibre $\mathbf{C}^{**} := \mathbf{C}\backslash\{\mathbf{0}, \mathbf{1}\}$. We shall distinguish these surfaces up to isomorphism and calculate their logarithmic Kodaira dimensions $\bar{\mathrm{k}}(\mathbf{X})$. For most of them $\bar{\mathrm{k}}(\mathbf{X}) = \mathbf{2}$, so they are of hyperbolic (or log-general) type. In **[Za 1]**, **[Za 3]** it is proved that

* for the original setting see, for instance, **[Ab Ha Ea]**

they are the only examples of acyclic surfaces of log-general type which support isotrivial families of curves with the base $\mathbf{C}$ (i.e. families with pairwise isomorphic generic fibres). Following **[Ra]** we use these surfaces in order to introduce exotic algebraic structures on affine spaces.

Main Theorem. *For any* $\mathbf{n} \geq 3$ *there exists a countable set of complex affine algebraic structures on* $\mathbf{R}^{2n}$ *which are pairwise biholomorphically nonequivalent.*

These structures can be distinguished in an algebraic sense, using the Strong Cancellation Theorem of Iitaka and Fujita **[Ii Fu]** . And by Strong Analytic Cancellation Theorem **1.10** they differ even in the analytic sense. Indeed, by the Iitaka-Fujita Theorem given an isomorphism $\mathbf{X} \times \mathbf{C}^n \to \mathbf{Y} \times \mathbf{C}^n$ the $\mathbf{C}^n$ can be cancelled if $\bar{k}(\mathbf{Y}) \geq 0$. By Theorem **1.10** below, given a biholomorphism $\mathbf{X} \times \mathbf{C}^n \to \mathbf{Y} \times \mathbf{C}^n$, where $\mathbf{X}$ and $\mathbf{Y}$ are quasiprojective, the $\mathbf{C}^n$ can be cancelled, giving an isomorphism $\mathbf{X} \to \mathbf{Y}$ if $\bar{k}(\mathbf{Y}) = \dim_{\mathbf{C}} \mathbf{Y}$, i.e. if $\mathbf{Y}$ is of hyperbolic type. The examples of non-cancellation for ($\mathbf{Q}$-acyclic) smooth affine surfaces with $\bar{k} = -\infty$ **[Da]** , **[tDi]** show that the assumptions of the first theorem are necessary, while for the second one this is unknown. I do not know also, whether there exist two different complex algebraic structures on $\mathbf{R}^{2n}$ which are analytically the same.

Furthermore, we show that for an acyclic surface $\mathbf{X}$ of hyperbolic type, $\mathbf{C}^n$ cannot be isomorphic to (and even cannot injectively dominate) a hypersurface of $\mathbf{X} \times \mathbf{C}^{n-1}$ (Theorem **2.4**). (In particular, 'exotic $\mathbf{C}^n$ constructed in such a way do not contain $\mathbf{C}^{n-1}$.) This is a generalization of a theorem in **[Za 3]** on the absence of simply connected curves in acyclic surfaces of general type.

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Remark. Recently A. Dimca, Sh. Kaliman and P. Russell have obtained new examples of exotic $\mathbf{C}^3$ – s, which are hypersurfaces in $\mathbf{C}^4$. For some of them $\bar{k} = 2$.

1. An analytic cancellation theorem

Let us first recall some known facts about holomorphic mappings into manifolds of hyperbolic type.

1.1. Definition [Ii 1]. *A nonsingular quasiprojective variety* $\mathbf{X}$ *is called a* ***manifold of hyperbolic type*** *iff its logarithmic Kodaira dimension* $\bar{\mathrm{k}}(\mathrm{X})$ *coincides with the dimension* $\mathbf{dim}_{\mathbf{C}}\mathbf{X}$.

1.2. Theorem. *Let* $\mathbf{X}$ *be a nonsingular quasiprojective variety and* $\mathbf{Y}$ *be a manifold of hyperbolic type. Then*

a) [Sa, Theorem 4.1] $\mathbf{Y}$ *is a volume hyperbolic complex manifold;*

b) [Sa, Proposition 4.2] *Every dominant holomorphic mapping* $\mathbf{X} \to \mathbf{Y}$ *is regular;*

c) [Ii 1, p. 182, Corollary] *Every dominant holomorphic mapping* $\mathbf{Y} \to \mathbf{Y}$ *is a biregular automorphism;*

d) [Ts] *The set* $\mathbf{Dom(X, Y)}$ *of all dominant holomorphic mappings* $\mathbf{X} \to \mathbf{Y}$ *is finite.*

In Corollaries **1.3 — 1.5** below we preserve the assumptions of Theorem **1.2**.

1.3. Corollary ([Ii 1, Theorem 6]; [Sa, Theorem 5.2]). *The group* $\mathbf{Aut\ Y}$ *of biregular automorphisms of* $\mathbf{Y}$ *is finite.*

1.4. Corollary. $\mathbf{Dom(X, Y)}$ *is an open and closed subspace of the space* $\mathbf{Hol(X, Y)}$ *of all holomorphic mappings* $\mathbf{X} \to \mathbf{Y}$*, endowed with the compact-open topology.*

1.5. Corollary. *Suppose that there exist mappings* $\varphi \in \mathbf{Dom(X, Y)}$ *and* $\psi \in \mathbf{Dom(Y, X)}$. *Then both* φ *and* ψ *are biregular isomorphisms.*

1.6. Definition [Ur]. *A complex manifold* $\mathbf{Y}$ ***belongs to class C*** *iff for any connected complex manifold* $\mathbf{Z}$ *and any holomorphic mapping* $\varphi\colon \mathbf{Y} \times \mathbf{Z} \to \mathbf{Y}$ *such that for some* $z_0 \in \mathbf{Z}$ *the mapping* $\varphi_{z_0} := \varphi|\, \mathrm{Y}\times\{z_0\}$ *belongs to the group* $\mathbf{Aut\ Y}$, *it follows that* $\varphi_z \equiv \varphi_{z_0}$ *for every* $z \in \mathbf{Z}$.

Let us recall that for manifolds of class C the cancellation theorem and the theorem of the uniqueness of a primary product-decomposition hold **[Ur]** .

1.10. Theorem. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be smooth irreducible quasiprojective manifolds of hyperbolic type. Let for some* $\mathbf{k}$ and $\mathbf{m} \geq 0$ *a biholomorphism* $\mathbf{\Phi}: \mathbf{X} \times \mathbf{C^k} \to \mathbf{Y} \times \mathbf{C^m}$ *be given. Then* $\mathbf{k} = \mathbf{m}$ *and there exists a unique biregular isomorphism* $\varphi: \mathbf{X} \to \mathbf{Y}$ *making the following diagram commutative:*

$$\begin{array}{ccc} X \times C^k & \xrightarrow{\Phi} & Y \times C^k \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi} & Y \end{array} .$$

In particular, $\mathbf{\Phi}$ *has a triangular form* $\mathbf{\Phi}(x,z) = (\varphi(x), \psi(x,z))$, *where* $(x,z) \in X \times C^k$ *and where for each* $x \in X$ *the mapping* $\psi_x := \psi \mid \{x\} \times C^k$ *belongs to the group* $\mathbf{Aut}\ \mathbf{C^k}$ *of biregular automorphisms of* $\mathbf{C^k}$.

Proof. By Theorem **1.2, a)** $\mathbf{X}$ and $\mathbf{Y}$ are volume hyperbolic manifolds. Hence by Corollary **1.9** $\dim_C X = \dim_C Y$ and $\mathbf{k} = \mathbf{m}$. Let us consider the holomorphic mapping $\varphi := \pi_Y \circ \Phi \mid X \times \{0_k\} : X \to Y$. We will show that φ is a dominant regular mapping.

The holomorphic mapping $f := \pi_Y \circ \Phi : X \times C^k \to Y$ is dominant, therefore $\dim \operatorname{Ker} df(u_0) = k$ for some $u_0 = (x_0 , z_0) \in X \times C^k$. Let $X' \subset X$ be an affine chart containing the point x_0 . There exists a regular mapping $\alpha : X' \to C^k$ such that $\alpha(x_0) = u_0$ and the graph $\Gamma(\alpha) \subset X' \times C^k$ is transversal to the subspace $\operatorname{Ker} df\,(u_0)$. Let $\tilde{\alpha} := \left(\mathrm{id}_{X'} , \alpha\right) : X' \hookrightarrow X' \times C^k$ be the embedding onto the graph $\Gamma(\alpha)$. It is easily seen that the mapping $\varphi_1 := f \circ \tilde{\alpha} : X' \to Y$ is dominant.

Consider a family of mappings $\varphi_t := f \circ \tilde{\alpha}_t$, where $\tilde{\alpha}_t := \left(\mathrm{id}_{X'} , t\alpha\right)$, $t \in C$. By Corollary **1.4** $\varphi_t \equiv \varphi_1$ for all $t \in C$, and by Theorem **1.2, b)** φ_1 is regular. Hence $\varphi = \varphi_0 = \varphi_1$ is a dominant regular mapping.

The same arguments applied to the mapping $\eta := \pi_X \circ \Phi^{-1} \mid Y \times \{0\} :$ $Y \to X$ show that η is a dominant regular mapping too. Therefore $\varphi : X \to Y$ is a biregular isomorphism (see Corollary **1.5**).

By Corollary **1.4** the mapping $\varphi_z := f \mid X \times \{z\} : X \to Y$, $z \in C^k$, does not depend on $\mathbf{z}$. Hence $\mathbf{\Phi}$ has a triangular form $\Phi(x,z) = (\varphi(x), \psi(x,z))$. Since $\mathbf{\Phi}$ is a biholomorphism the mapping $\psi_x : C^k \to C^k$ is biholomorphic for all $x \in C^k$. This completes the proof.