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SŁAWOMIR CYNK

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REPRESENTATIONS OF NASH FUNCTIONS

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Introduction.

The aim of this paper is to characterize Nash functions of m complex variables in term of rational functions of $m+1$ variables.

Using the notation introduced in Chapter I of the paper our main result (Theorem III.2.1) may be formulated as follows:

Let K be a compact, rationally convex subset of $\mathbb{C}^m$. A function

$$f\colon K \longrightarrow \mathbb{C}$$

extends to a Nash function in a neighborhood of K if and only if there is a rational function $R \in \mathbb{C}(z,w)$, holomorphic in neighborhood of $K \times T$ (where T denotes the unit circle in $\mathbb{C}$), such that

$$f(z) = \int_T R(z,w)dw \qquad \text{for} \qquad z \in K.$$

The paper is organized as follows:

Chapter I and II are of preparatory nature. In Chapter I we study the class of rationally convex compact sets. As this class is essential in our further considerations, we give detailed proves of all theorems that we shall use later.

The aim of Chapter II is to characterize Nash functions in terms of a special class of Nash functions – called simple Nash functions (Lemma II.3.2). This Lemma (in the case of m=1) was earlier obtained in [C–T]. In [D–L] similar result ("in local situation") was proved.

Chapter III contains main results of our paper.

Our result were inspired by [C–T] and [D–L]. We apply some methods used in these papers.

CHAPTER I
Rationally Convex Compact Sets

1. Rational Functions . In this section we present some basic properties of rational functions We shall need them in further sections of this paper.

Let us start with the definition of rational function on an algebraic subset V of $\mathbb{C}^m$.

DEFINITION 1. *The ring of rational functions of the set V, denoted by $\mathbb{C}(V)$, is the full ring of fraction of the coordinate ring R_V of the set V. An element of the ring $\mathbb{C}(V)$, is called a rational function on V.*

Let f be an arbitrary rational function on V. According to the definition there exist two regular functions P, Q on V such that:

1. Q is not a zero-divisor in the ring R_V (in other words Q is not identically equal 0 on any irreducible component of the algebraic set V),
2. $f = \dfrac{P}{Q}$.

DEFINITION 2. *A rational function $f = \dfrac{P}{Q}$ is said to be holomorphic at point $a \in V$ iff there exists a germ $g \in \mathcal{O}_a(V)$ of holomorphic function at the point a such that $g \cdot Q = P$.*

Let us notice that the germ g is uniquely determined (does not depend on the choice of regular functions P i Q). The set of point at which a rational function is holomorphic is an open and dense (in euclidean topology) subset of the set V.

The following theorem yields more precise characterization that set.

THEOREM 1. *Let f be a rational function of V. The set of all points at which the function f is not holomorphic, is a nowhere-dense algebraic subset of V.*

Proof. There exist regular functions $P, Q \in R_V$ such that $f = \dfrac{P}{Q}$ and the function Q does not vanish at any irreducible component of the set V. The set

$$X_0 := \{(x, w) \in V \times \mathbb{P}^1(\mathbb{C}) : Q(x) \neq 0 \quad \text{and} \quad w \cdot Q(x) = P(x)\}$$

is a constructible subset and the set $X := \overline{X_0}$ is an algebraic subset of $V \times \mathbb{P}^1(\mathbb{C})$. Moreover $X_1 := X \cap (\mathbb{C}^m \times \mathbb{C})$ is an algebraic subset of $\mathbb{C}^m \times \mathbb{C}$.

Let us assume that the function f is holomorphic at a point $a \in V$. There exists a germ $g \in \mathcal{O}_a(V)$ of holomorphic function such that $g \cdot Q = P$. In this situation

$$X \cap (\{a\} \times \mathbb{P}^1) = \{(a, g(a))\}.$$

Choose an arbitrary holomorphic germ $g_1 \in \mathcal{O}_a(\mathbb{C}^m)$ such that $g_1|_V = g$. The holomorphic germ $h \in \mathcal{O}_{(a,f(a))}(\mathbb{C}^m \times \mathbb{C})$ defined by the formula

$$h(z_1, \dots, z_m, z_{m+1}) := z_{m+1} - g_1(z_1, \dots, z_m),$$

is an element of the ideal of the germ of the analytic set X_1 at the point $(a, g(a))$.

Using the Serre Lemma (on polynomial generators) ([L], VII.15.3., p.337) we conclude that there exist polynomials $P_1, \dots, P_k \in I(X_1)$ ($I(X_1)$ denotes the ideal of the algebraic set X_1) and germs of holomorphic functions $g_1, \dots, g_k \in \mathcal{O}_{(a,f(a))}(\mathbb{C}^m \times \mathbb{C})$ such that $g = g_1 P_1 + \dots, g_k P_k$. Differentiating the above equality we observe that for at least one index $i = 1, \dots, k$ we have $\frac{\partial P_i}{\partial z_{m+1}}(a, f(a)) \neq 0$.

Denoting by W the set of all points at which the function f is not holomorphic we state that

$$\begin{aligned} W_1 := \Big\{a \in V : \exists a_{m+1} \in \mathbb{C}: (a, a_{m+1}) \in X \\ \text{and} \quad \forall F \in I(X_1) \quad \frac{\partial F}{\partial z_{m+1}}(a, a_{m+1}) = 0\Big\} \\ \cup \Big\{a \in V: (a, \infty) \in X\Big\} \cup \Big\{a \in V: \#\left(X \cap (\{a\} \times \mathbb{P}^1)\right) \geq 2\Big\} \subset W. \end{aligned}$$

We shall prove that

$$W_1 = W.$$

Suppose, on the contrary, that $a \in W \setminus W_1$.

From the definition of W_1 we have

$$X \cap (\{a\}) \times \mathbb{P}^1 = \{(a, a_{m+1})\}, \qquad a_{m+1} \in \mathbb{C}.$$

Moreover there is a polynomial $F \in I(X_1)$ such that $\frac{\partial F}{\partial z_{m+1}}(a, a_{m+1}) \neq 0$. By the implicit function theorem there exist an open neighborhood U of $a \in \mathbb{C}^m$, a real number $r > 0$ and a holomorphic function $\phi: U \longrightarrow a_{m+1} + \Delta(r)$ (where $\Delta(r) := \{z \in \mathbb{C}: |z| < r\}$) such that $F^{-1}(0) \cap \left(U \times \left(a_{m+1} + \bar{\Delta}(r)\right)\right) = \phi$.

As the natural projection

$$\pi: X \ni (x_1, \ldots, x_m, x_{m+1}) \mapsto (x_1, \ldots, x_m) \in V$$

is a proper mapping we may assume (if necessary – after suitable decreasing of U and r) that

$$X \cap (U \times \mathbb{C}) \subset \phi|_{(V \times U)}.$$

From the latest equality we can deduce that for any point $z \in V \times U$ such that $Q(z) \neq 0$ we have $\dfrac{P(z)}{Q(z)} = \phi(z)$.

Since the set $\{z \in V \times U: Q(z) \neq 0\}$ is dense in $V \times U$ we have

$$P(z) = Q(z) \cdot \phi(z) \qquad \text{for any} \qquad z \in V \times U,$$

and this means that the rational function f is holomorphic at the point a. We obtain a contradiction which proves that $W = W_1$.

Let us notice that $W = W_1$ is an algebraically constructible set ([K] Th.III.11.1.; [Ł], VII.8.3.— the Chevalley Theorem), and hence — since it is closed — an algebraic set (cf. [Ł], VII.8.3., p. 291—295). The proof is completed □

Let Ω be an open subset of $\mathbb{C}^m$. We shall denote by $\mathcal{R}(\Omega)$ the space of all holomorphic functions on Ω which are restrictions to the set Ω of rational functions. Let us notice that a function $f: \Omega \to \mathbb{C}$ belongs to $\mathcal{R}(\Omega)$ if and only if there exist polynomials $P, Q: \mathbb{C}^m \to \mathbb{C}$ such that $Q^{-1}(0) \cap \Omega = \emptyset$ and $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in \Omega$. If the polynomials P, Q are relatively prime then their are uniquely determined (up to a constant factor).

Let K be a fixed compact subset of $\mathbb{C}^m$. Denote

$$\mathcal{O}(K) := \{f: K \to \mathbb{C}: \quad \text{there exist an open neighborhood } V \text{ of } K \text{ and a function } \tilde{f} \in \mathcal{O}(V) \text{ such that } f = \tilde{f}|_K\}.$$

An extension of a function from the class $\mathcal{O}(K)$ to an open neighborhood of K is not uniquely determined.

In the same way as $\mathcal{O}(K)$ we define the class $\mathcal{R}(K)$.

Let us observe that a function $f: K \to \mathbb{C}$ belongs to the class $\mathcal{R}(K)$ if and only if there exist polynomials $P, Q: \mathbb{C}^m \to \mathbb{C}$ such that $Q^{-1}(0) \cap K = \emptyset$ and $f(z) = \dfrac{P(z)}{Q(z)}$ for $z \in K$. Polynomials P and Q are not (in general) uniquely determined (even up to a constant factor).