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ERIC AMAR

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A PROBLEM OF IDEALS

Eric Amar

Recently U. Cegrell [2] proved the following result:

Theorem. *Let $\mathbb{B}$ be the unit ball of $\mathbb{C}^n$ and $f_1 \in A(\mathbb{B})$, $f_2 \in H^\infty(\mathbb{B})$such that $\forall z \in \mathbb{B}$, $|f_1(z)| + |f_2(z)| \geq \delta$ then there are two functions g_1, g_2 in $H^\infty(\mathbb{B})$ such that: $f_1 g_1 + f_2 g_2 = 1$ in $\mathbb{B}$.*

where, as usual, if Ω is a domain in $\mathbb{C}^n$, $A(\Omega)$ is the algebra of all holomorphic functions in Ω continuous up to the boundary, $A^k(\Omega)$ is the algebra of all holomorphic functions in Ω, $\mathcal{C}^k$ up to $\partial\Omega$ and $H^\infty(\Omega)$ is the algebra of all holomorphic and bounded functions in Ω.

This means, in this special case, that the Corona is true. He uses a nice analysis of pic functions and representing measures in the ball, of independant interest.

The aim of this note is to give a very simple proof of this theorem which is also more general.

In order to state the result, let me give the following definition:

Definition. *We say that the bounded pseudo-convex domain in $\mathbb{C}^n$ has the L_q^∞ property if: for any $(0, q)$ form ω in $\mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$, there is a $(0, q-1)$ form u in $\mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$ such that: $\bar{\partial} u = \omega$.*

As usual, a (0,0) form is just a function.

There are many examples of such domains: the strictly pseudo-convex ones [6], the polydiscs [8], the ellipsoïds [4], [11], the domains of finite type in $\mathbb{C}^2$ [3], [5].

We shall prove the following:

Theorem 1. *Let Ω be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the L_1^∞ condition and let $f_1 \in A(\Omega)$, $f_2 \in H^\infty(\Omega)$ such that $\forall z \in \Omega$, $|f_1(z)|+|f_2(z)| \geq \delta$ then there are two functions g_1, g_2 in $H^\infty(\Omega)$ such that: $f_1 g_1 + f_2 g_2 = 1$ in Ω.*

Proof:
because f_1 is continuous up to $\partial\Omega$, it is easy to make a function $\chi \in \mathcal{C}^\infty(\overline{\Omega})$ such that:

$$\chi = \begin{cases} 1 & in \ \{|f_1| > \delta/2\} \\ 0 & in \ \{|f_1| < \delta/4\} \end{cases}$$

Now let $\omega := \frac{\overline{\partial}\chi}{f_1 f_2}$, then $\omega \in \mathcal{C}^\infty(\Omega) \cap L^\infty(\Omega)$ because on the set where $\overline{\partial}\chi \neq 0$, $|f_1 f_2| > \delta^2/16$. Moreover, $\overline{\partial}\omega = 0$ in Ω, hence, by the L_1^∞ condition, there is a $u \in L^\infty(\Omega)$ such that: $\overline{\partial}u = \omega$.

Let us define

$$g_1 := \frac{\chi}{f_1} - u f_2 \text{ and } g_2 := \frac{1-\chi}{f_2} + u f_1;$$

then we get:

$$\overline{\partial}g_1 = 0,\ \overline{\partial}g_2 = 0$$

hence these functions are holomorphic in Ω and:

$$f_1 g_1 + f_2 g_2 = 1.$$

Moreover the g_i's are easily seen to be bounded in Ω, hence the theorem. ∎

Now using the Koszul's Complex method as in [9], it is easy to prove, usingexactly the same lines the:

Theorem 2. *Let Ω be a pseudo-convex bounded domain in $\mathbb{C}^n$ verifying the L_q^∞ condition for $q \leq p-1$ and let $f_1, .., f_{p-1} \in A(\Omega)$, $f_p \in H^\infty(\Omega)$ such that $\forall z \in \mathbb{B}$ $\sum_{i=1}^p |f_i(z)| \geq \delta$; then there are p functions $g_1, ..., g_p$ in $H^\infty(\Omega)$ such that: $\sum_i f_i g_i = 1$ in Ω.*

Now let us define the $\mathcal{C}_p^k$ property for a pseudo-convex bounded domain in an analogous way:

Definition. *We say that the bounded pseudo-convex domain in* $\mathbb{C}^n$ *has the* $\mathcal{C}_q^k$ *property if: for any* $(0,q)$ *form* ω *in* $C^k(\overline{\Omega})$, *there is a* $(0,q-1)$ *form* u *in* $C^k(\overline{\Omega})$ *such that:* $\overline{\partial}u = \omega$.

For k finite, the domains listed above with the L_q^∞ property have the $\mathcal{C}_q^k$ property too. For $k = \infty$ a very famous theorem by J.J. Kohn [10] says that all pseudo-convex bounded domains with smooth boundary has the $\mathcal{C}_q^\infty$ property.

The same way has above, we can show:

Theorem 3. *Let* Ω *be a pseudo-convex bounded domain in* $\mathbb{C}^n$ *verifying the* $\mathcal{C}_q^k$ *condition for* $q \leq p-1$ *and let* $f_1, .., f_p \in A^k(\Omega)$, *such that* $\forall z \in \Omega \sum_{i=1}^p |f_i(z)| \geq \delta$; *then there are* p *functions* $g_1, ..., g_p$ *in* $A^k(\Omega)$ *such that:* $\sum_i f_i g_i = 1$ *in* Ω.

As a classical corollary we get:

Corollary. *Let* Ω *be a pseudo-convex bounded domain in* $\mathbb{C}^n$ *verifying the* $\mathcal{C}_q^k$ *condition for* $q \leq n$, *then the spectrum of the algebra* $A^k(\Omega)$ *is* $\overline{\Omega}$.

In the case $k = \infty$, M. Catlin [1] and M. Hakim and N. Sibony [7] already proved this result, the method they used is also a division method but slightly different and their method cannot give theorem 1 and 2 here.

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Amar Eric
Université Bordeaux I
351 Cours de la Libération
33405 Talence, France