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Astérisque, tome 218 (1993), p. 111-144

<http://www.numdam.org/item?id=AST_1993__218__111_0>

SCHUR QUADRICS, CUBIC SURFACES AND RANK 2 VECTOR BUNDLES OVER THE PROJECTIVE PLANE

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Let $\Sigma \subset P^3$ be a smooth cubic surface. It is known that S contains 27 lines. Out of these lines one can form 36 *Schläfli double - sixes* i.e., collections $\{l_1, ..., l_6\}, \{l'_1, ..., l'_6\}$ of 12 lines such that each l_i meets only l'_j, $j \neq i$ and does not meet $l_j, j \neq i$, see n.0.1 below. In 1881 F. Schur proved **[S]** that any double - six gives rise to a certain quadric Q , called *Schur quadric* which is characterized as follows: for any i the lines l_i and l'_i are orthogonal with respect to (the quadratic form defining) Q.

The aim of the present paper is to relate Schur's construction to the theory of vector bundles on P^2 and to generalize this construction along the lines of the said theory.

Let us describe the vector bundle interpretation of the Schur quadric. Note that the first six lines $\{l_1, ..., l_6\}$ of a double - six on Σ define a blow-down $\pi : \Sigma \rightarrow P^2$ which takes the lines l_i into some points $p_i \in P^2$. These points are in general position i.e. no three of them lie on a line. Let $\check{P}^2$ be the dual projective plane and $H_i \subset \check{P}^2$ be the lines corresponding to p_i. The union $\mathcal{H}$ of these lines is a divisor with normal crossing in $\check{P}^2$. Let $E(\mathcal{H}) = \Omega^1_{\check{P}_2}(\log \mathcal{H})$ be the corresponding vector bundle (locally free sheaf) of logarithmic 1-forms on $\check{P}^2$. The twisted bundle $E = E(\mathcal{H})(-2)$ is a stable rank 2 bundle on $\check{P}^2$ with Chern classes $c_1 = -1, c_2 = 4$ (see **[DK]**). For such bundles K.Hulek **[Hu1]** has defined the notion of a *jumping line of the second kind* (shortly JLSK). This is a line $l \subset \check{P}^2$ such that the restriction of E to the first infinitesimal neigborhood $l^{(1)}$ of l is not isomorphic to $\mathcal{O}_{l^{(1)}} \oplus \mathcal{O}_{l^{(1)}}(-1)$. Hulek has shown that such lines form a

* Research of both authors supported in part by NSF.

curve $C(E)$ in the projective plane of lines in $\check{P}^2$ i.e. in P^2. Now the result is as follows.

Theorem 1. *The space P^3 containing the cubic surface Σ is naturally identified with the projectivization of $H^1(\check{P}^2, E(-1))^*$. Under this identification the Schur quadric Q becomes dual to the zero locus of the quadratic form given by the cup-product*

$$H^1(\check{P}^2, E(-1)) \otimes H^1(\check{P}^2, E(-1)) \to H^2(\check{P}^2, \bigwedge^2(E(-1))) = H^2(\check{P}^2, \mathcal{O}(-3)) = \mathbf{C}.$$

The intersection $\Sigma \cap Q$ is mapped, under the projection $\pi : \Sigma \to P^2$, to the curve of JLSK $C(E)$.

More generally, the whole theory of Hulek **[Hu1]** of rank 2 vector bundles on P^2 with odd c_1 can be given a "geometric" interpretation involving some natural generalizations of cubic surfaces, double - sixes and Schur quadrics. This is done in §2 of the paper. This interpretation implies Theorem 1.

The outline of the paper is as follows. In §0 we recall some known (and less known) facts about cubic surfaces and Schur quadrics. In §1 we give a short overview of Hulek's theory of monads corresponding to vector bundles with $c_1 = -1$. In §2 we give an interpretation of Hulek's theory mentioned above. In §3 we consider bundles of logarithmic 1-forms corresponding to arrangements of $2d$ lines in P^2 in general position. The main result of this section is that all these bundles satisfy certain condition of Σ - genericity in the sense defined in §2, which makes working with bundles satisfying this condition easier. Finally, in §4 we consider various examples of the previous constructions corresponding to some special types of vector bundles.

§0. Cubic surfaces.

0.1. Here we recall some standard known facts about cubic surfaces. All the proofs can be found either in **[H]**, Ch.V, §4 or in **[M]** or can be easily reconstructed by the reader. Let $p_1, ..., p_6$ be six distinct points in the projective plane P^2. Assume that no three of these points lie on a line. Denote by Z the union of the points p_i and by $\mathcal{J}_Z \subset \mathcal{O}_{P(V)}$ the sheaf of ideals of Z. The linear system $P\big(H^0(\mathcal{J}_Z(3))\big)$ of cubic curves through Z is of dimension 3 and defines a rational map

$$f : P^2 \to P(H^0(\mathcal{J}_Z(3)^*) = P^3$$

whose image is a cubic surface, denoted Σ. The rational map f comes from a regular map $f' : \mathrm{Bl}_Z(P^2) \to P^3$ where $\mathrm{Bl}_Z(P^2)$ is the blow up of Z. Let $\pi : \mathrm{Bl}_Z(P^2) \to P^2$ be the projection. If we further assume that the points p_i do not lie on a conic then f' is an isomorphism and Σ is nonsingular. If p_i do lie on a conic then Σ is singular and f' blows down this conic to a singular point of Σ.

Suppose Σ is nonsingular. Then Σ has 27 lines on it. They can be grouped into three subsets:

$$\{l_1, ..., l_6\},\ \{l'_1, ..., l'_6\},\ \{m_{ij},\ 1 \le i < j \le 6\}. \tag{0.1}$$

The lines l_i are the images under f' of the exceptional lines $\pi^{-1}(p_i)$. The lines l'_i are images under f' of proper transforms of the conics $C_i \subset P^2$ passing through $Z - \{p_i\}$. Finally the lines m_{ij} are images of the proper transforms of the lines $< p_i, p_j >$ joining the points p_i and p_j.

The first two groups of lines form a *double - six* which means that

$$l_i \cap l_j = \emptyset, \quad l'_i \cap l'_j = \emptyset, \quad l_i \cap l'_j \ne \emptyset \qquad \text{iff} \quad i \ne j. \tag{0.2}$$

Every set of 6 disjoint lines on Σ can be included in a unique double - six from which Σ can be reconstructed uniquely. There are 36 double - sixes of Σ. Every double - six defines two regular birational maps $\pi_1 : \Sigma \to P^2$, $\pi_2 : \Sigma \to P^2$, each blowing down one of the two sixes (sixtuples of disjoint lines) of the double - six.

The birational map $\pi_2 \circ \pi_1^{-1} : P^2 \to P^2$ is given by the linear system of quintics with double points at p_i. The two collections of 6 points in P^2 given by $\{\pi_1(l_i)\}$ and $\{\pi_2(l'_i)\}$ are associated to each other in the sense of Coble (cf.[**DO**],[**DK**]).

0.2. Here we shall discuss somewhat less known facts about the determinantal representation of a cubic surface [**B**]. A modern treatment of this can be found in [**G**],[**Gi**]. Consider the homogeneous ideal of the subscheme Z i.e.

$$I_Z = \bigoplus_{n \geq 0} H^0(P^2, \mathcal{J}_Z(n)) \tag{0.3}$$

in the graded ring $R = \mathbf{C}\ [T_0, T_1, T_2]$. It is easy to see that the ring R/I_Z is Cohen - Macaulay hence of homological dimension 1. Any four linearly independent cubic forms vanishing on Z represent a minimal set of generators of I_Z. According to the Hilbert-Burch theorem (see [**No**],7.5) the ideal I_Z is generated by the maximal minors of some 3×4 matrix of homogeneous linear forms. In other words, we have a resolution

$$0 \to R(-4)^3 \to R(-3)^4 \to I_Z \to 0.$$

This resolution gives the resolution of the sheaf $\mathcal{J}_Z(3)$:

$$0 \to \mathcal{O}_{P(V)}(-1)^3 \to \mathcal{O}_{P(V)}^4 \to \mathcal{J}_Z(3) \to 0.$$

We can rewrite this resolution in the form

$$0 \to \mathcal{O}_{P^2}(-1) \otimes I^* \xrightarrow{\gamma} \mathcal{O}_{P^2} \otimes L^* \to \mathcal{J}_Z(3) \to 0 \tag{0.4}$$

where vector spaces I^* and L^* of respective dimensions 3 and 4 are defined intrinsically as follows:

$$L^* = H^0(P^2, \mathcal{J}_Z(3)); \tag{0.5}$$

$$I^* = \mathrm{Ker}\{H^0(P^2, \mathcal{O}(1) \otimes L^*) \to H^0(P^2, \mathcal{J}_Z(4))\}. \tag{0.6}$$

Note that one can also obtain (0.4) from the Beilinson spectral sequence applied to the sheaf $\mathcal{J}_Z(3)$. It gives also an isomorphism

$$I^* \cong H^1(P^2, \mathcal{J}_Z(1)).$$