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DECOMPOSITION OF SPECTRAL COVERS

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1 Introduction

To a vector bundle $E \to X$ and an endomorphism $\varphi : E \to E$ one associates a spectral cover $\pi : \widetilde{X} \to X$, whose fibers $\pi^{-1}(x)$, $x \in X$, are given by the eigenvalues of φ_x. If φ is, more generally, a K-valued endomorphism (a "Higgs bundle") $\varphi : E \to E \otimes K$, where K is a line bundle on X, we still get a cover $\pi : \widetilde{X} \to X$, but now $\widetilde{X}$ is contained in the total space $|K|$ of K, since the eigenvalues live in K. The eigenspaces of φ give a sheaf L on $\widetilde{X}$, which is a line bundle if φ is regular [BNR,B]. (Even more generally, K can be allowed to be a vector bundle on X, as long as a symmetry condition (trivial in case K is a line bundle) is imposed on φ: the case where K is the cotangent bundle of X arises in [S]. In this work we will consider only the case of a line bundle K.) One way to construct these objects is to let $\pi_K : |K| \to X$ denote the natural projection, and let τ be the tautological section of $\pi_K^* K$. Then $\pi_K^* \varphi - \tau$ is a $\pi_K^* K$-valued endomorphism of $\pi_K^* E$. Now L is the cokernel of $\pi_K^* \varphi - \tau$, considered as a sheaf on its support $\widetilde{X} := Supp(L) \subset |K|$.

This situation arises frequently in the study of completely integrable Hamiltonian systems on a manifold M which can be written as a Lax equation depending on parameters [AvM,B,G,H,K]; here X is the parameter space, often the affine line or $\mathbf{P}^1$, and the flow of the system is linearized on the Picard variety $Pic\widetilde{X}$ (or the Jacobian, when X is a curve). The linearization map typically gives an isogeny from the Liouville tori of the completely integrable system to (an Abelian subvariety of) $Pic\widetilde{X}$, by sending a point of M where the Lax equation is regular to the eigen line bundle computed at that point.

The vector bundle $E \to X$ often has G-structure, where G is some reductive Lie group. In other words, E is associated to a principal G-bundle $\mathcal{V} \to X$ via a representation $\rho : G \to GL(V)$ of G. The endomorphism φ then becomes a section of $\mathbf{ad}\mathcal{V} \otimes K$, where $\mathbf{ad}\mathcal{V}$ is the associated bundle of Lie algebras $\mathcal{V} \times_G \mathbf{g}$. In [AvM], Adler and van Moerbeke raised the question of the dependence of the resulting cover $\widetilde{X}_\rho$ on the representation ρ. If the situation comes from a completely integrable system as above, then the Liouville torus, which depends on the differential equation but not on the particular Lax equations or on the representation ρ, should occur, up to isogeny, as a subvariety of $Pic\widetilde{X}_\rho$, for all ρ. One may therefore expect to find a natural, Prym-type subvariety of each $Pic\widetilde{X}_\rho$, together with correspondences between pairs $\widetilde{X}_\rho$, $\widetilde{X}_{\rho'}$ whose images in the Picard varieties should be isogenous to this generalized Prym. More generally, one may wish to describe all correspondences acting on each $\widetilde{X}_\rho$ (or between pairs) over the base X, and to find the isogeny decomposition of $Pic\widetilde{X}_\rho$

into isotypic pieces under this action. One of these isotypic pieces should be common to all $\widetilde{X}_\rho$, and this would be the generalized Prym.

Several special cases of this situation, arising from orthogonal groups, are well known in Prym theory, e.g. Recillas' trigonal construction [R], my tetragonal construction [D1,D2], and Pantazis' bigonal construction [P]. The case of the exceptional group G_2 is discussed in [KP]. Other examples, related to the geometry of families of Del Pezzo surfaces, are given in [K]. In that work, Kanev gives a solution of Adler-van Moerbeke's question, under a few hypotheses: the base X is $\mathbf{P}^1$, the principal bundle $\mathcal{V}$ is trivial, the Lie algebra $\mathbf{g}$ is simple of type A_n, D_n, or E_n. Under these assumptions he constructs, for each $\widetilde{X}_\rho$, a Prym-Tyurin variety $Prym(\widetilde{X}_\rho/X) \subset Jac(\widetilde{X}_\rho)$ and a correspondence whose image is $Prym(\widetilde{X}_\rho/X)$. The Prym-Tyurin varieties for different representations are isogenous, and even isomorphic if both representations are minuscule.

The purpose of this work is to analyze the decomposition of the Picard varieties of general spectral covers for a reductive group G. We will show (Theorem 8.1) that there is a distinguished isotypic component of $Pic\widetilde{X}_\rho$, corresponding to the reflection representation Λ of the Weyl group W. When G is one of the classical simple groups, this is the unique piece common to $Pic\widetilde{X}_\rho$ for all non-trivial representations ρ of G. For some exceptional groups the uniqueness fails, as we see in Sections 10,11.

Our approach throughout is based on the observation that the geometry of the spectral covers reflects not so much the representations of G as those of its Weyl group W. Various questions about a spectral cover $\widetilde{X}_\rho$ simplify considerably when the emphasis is placed on the action of W rather than on the way $\widetilde{X}_\rho$ sits inside K. Here is what we do in more detail:

The spectral covers $\widetilde{X}_\rho$ decompose into subcovers $\widetilde{X}_\lambda$, indexed by W-orbits of weights λ. There are infinitely many distinct covers $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$, but they fall into only a finite number (2^r, where $r = rank_{ss}(G)$) of birational classes, cf. lemma (3.3). In section 2 we construct an abstract W-Galois cover $\widetilde{X} \to X$ which dominates all $\widetilde{X}_\lambda$. In good cases, points of $\widetilde{X}$ over $x \in X$ parametrize chambers in the dual of the unique Cartan subalgebra $\mathbf{t}(\varphi(x))$ containing $\varphi(x)$, so we call $\widetilde{X} \to X$ the *cameral* cover. With very few exceptions (listed in (4.3)), the spectral covers $\widetilde{X}_\lambda$ are forced to be singular as soon as X contains a compact curve, while the cameral cover $\widetilde{X}$ and its quotients by the parabolic subgroups serve as natural desingularizations, as long as the endomorphism φ remains regular. For example, this happens for $\mathbf{g} = \mathbf{so}(2n)$ and *any* non-trivial representation. (For the standard, $2n$-dimensional representation of $\mathbf{so}(2n)$, Hitchin notes these accidental singularities in [H], and attributes them to the vanishing of the Pfaffian.) In particular, it is unrealistic to hope that the eigensheaf L will "generically" be a line bundle on $\widetilde{X}_\rho$ or $\widetilde{X}_\lambda$: In typical situations we get torsion free sheaves on $\widetilde{X}_\lambda$, which come from line bundles on the cameral $\widetilde{X}$. (The original situation, where $G = GL(n)$ and ρ is the standard representation, is thus quite atypical!)

The ring of natural correspondences on $\widetilde{X}_\lambda$ is described in §6 in terms of the Weyl group W and the parabolic subgroup W_P determined by λ. The question of decomposing the spectral Picards is translated to decomposition of the permutation representation $\mathbf{Z}[W/W_p]$ as W-module. Some general results, based on Springer's representation and the work of [BM], are reviewed in Section 9. These results clarify the general form of the decomposition, but do not seem to imply the uniqueness of the common component. We thus work out the uniqueness for classical groups, and the non-uniqueness for some exceptional groups, by direct computations, in Sections 8, 10 and 11.

In this group-theoretic context, actually writing down the decomposition in any given case is very easy. In §12, we write down some formulas for the projection of a spectral Picard onto any generalized Prym. In the case of the projection to the distinguished Prym we recover Kanev's formulas (with minor modifications, which we explain). Kanev's construction, which is very geometric, is motivated by the interpretation of certain Weyl groups as symmetries of line configurations on rational surfaces. Our point is that similar formulas work much more generally, and require only elementary group theory. J.Y. Merindol informed me, during the Orsay conference, that he has also obtained projection formulas (onto the distinguished Prym) for arbitrary reductive groups, removing Kanev's restriction to "simply laced" groups, of types A_n, D_n, E_n.

For our purpose in this paper, we can take G to be any complex reductive group, but the resulting spectral and cameral covers depend only on the semisimple part G_{ss} of G, as does the distinguished Prym. There is however a more natural subvariety of $Pic\widetilde{X}$, consisting up to isogeny of $Prym(\widetilde{X})$ together with a number (equal to the dimension of the center of G) of copies of $PicX$. This corresponds to the reflection representation of W on the weights of G, which decomposes up to isogeny into the weights of G_{ss} and a trivial representation. In a sequel to this work [D3] we will describe this enlarged Prym in terms of W-equivariant bundles on $\widetilde{X}$, and interpret it as a moduli space of generalized Higgs bundles on X with given spectral invariants. Combined with work of Markman on the existence of Poisson structures [M], this leads to an algebraically completely integrable Hamiltonian system, generalizing those of Hitchin, Jacobi-Mumford-Beauville [B], and so on. The construction extends