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SESHADRI CONSTANTS ON SMOOTH SURFACES

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Introduction

Let X be a smooth complex projective variety of dimension n, and let L be a numerically effective line bundle on X. Following Demailly [**De2**], one defines the *Seshadri constant* of L at a point $x \in X$ to be the real number

$$\epsilon(L, x) = \inf_{C \ni x} \frac{L \cdot C}{m_x(C)} \quad ,$$

where the infimum is taken over all irreducible curves C passing through x, and $m_x(C)$ is the multiplicity of C at x. It is profitable to view $\epsilon(L, x)$ as a local measure of how positive L is at x. For example if L is very ample, then $\epsilon(L, x) \geq 1$; on a surface X the same is true more generally if $L = \mathcal{O}_X(D)$ for an ample effective divisor $D \ni x$ which is smooth at x. In general, if $f : Bl_x(X) \longrightarrow X$ denotes the blowing up of X at x and $E = f^{-1}(x)$ is the exceptional divisor, then for $\epsilon > 0$ the $\mathbb{R}$-divisor $f^*L - \epsilon \cdot E$ is nef if and only if $\epsilon \leq \epsilon(L, x)$. (Consult [**De2**, §6] for other interpretations.) Similarly, one defines the global Seshadri constant

$$\epsilon(L) = \inf_{x \in X} \epsilon(L, x).$$

Thus Seshadri's criterion for ampleness states that $\epsilon(L) > 0$ if and only if L is ample.

Recent interest in Seshadri constants stems from the fact that they govern a simple method for producing sections of adjoint bundles $K_X + kL$ (c.f. [**De2**, (6.8)]). In brief, by means of vanishing theorems on the blow-up $Bl_x(X)$, a lower bound on $\epsilon(L, x)$ yields an explicit value of k such that $K_X + kL$ has a section which is non-zero at x (see (3.4) below). We shall see in §3 that Seshadri

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constants alone cannot account for the known results on global generation and very ampleness of adjoint bundles ([**Rdr**], [**De1**], [**EL**]). However they remain very interesting in their own right as measures of local positivity. The subtlety of these invariants is reflected in the fact, pointed out by Demailly, that they are already rather difficult to compute on surfaces.

The purpose of this note is to study Seshadri constants in this first non-trivial case, when X is a smooth projective surface. One might anticipate that in general $\epsilon(L,x)$ could become small on fairly arbitrary algebraic subsets of X. Somewhat surprisingly, our main result shows that this is not the case:

THEOREM. *Let L be an ample line bundle on a smooth complex projective surface X. Then $\epsilon(L,x) \geq 1$ for all except perhaps countably many points $x \in X$, and moreover if $c_1(L)^2 > 1$, then the set of exceptional points is in fact finite. More generally, given an integer $e > 1$, suppose that*

$c_1(L)^2 \geq 2e^2 - 2e + 1$ *and* $c_1(L) \cdot \Gamma \geq e$ *for every irreducible curve* $\Gamma \subset X$.

Then $\epsilon(L,x) \geq e$ for all but finitely many $x \in X$.

On the other hand, simple examples (constructed by Miranda) show that $\epsilon(L,x)$ can take on arbitrarily small values at isolated points. We hope that this gives some sense of the kind of picture one might hope for in higher dimensions.

The proof of the theorem is completely elementary, the essential point being simply to view the question variationally. Specifically, suppose that L is an ample line bundle, and $C = C_0 \subset X$ is a curve with $m = m_x(C) > C \cdot L$ for some point $x = x_0 \in C$. By combining a simple computation in deformation theory (§1) with the Hodge index theorem, we show that (C,x) cannot move in a non-trivial one-parameter family (C_t, x_t) with $m_{x_t}(C_t) \geq m$ for all t. In other words, pairs (C,x) forcing $\epsilon(L,x) < 1$ are rigid, and the first statement of the Theorem follows at once. We were inspired in this argument by work of G. Xu [**Xu**], who uses related but much more elaborate calculations to study geometric genera of subvarieties of general hypersurfaces in projective space. We present some examples and open questions in §3.

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§1. Deformations of Singular Curves on a Surface

This section is devoted to a proof, in the spirit of [**Xu**], of an elementary lemma concerning the deformation theory of singular curves on a surface. While the result in question is certainly well known in the folklore, we include an argument here for lack of a suitable reference and for the convenience of the reader.

We consider the following situation. X is a smooth complex projective surface, and we suppose given a one-parameter family

$$\{ C_t \ni x_t \}_{t \in \Delta}$$

consisting of curves $C_t \subset X$ plus a point $x_t \in C_t$, parametrized by a smooth curve or small disk Δ. Setting $C = C_0$ and $x = x_0$ for $0 \in \Delta$, the deformation determines a Kodaira-Spencer map

$$\rho : T_0\Delta \longrightarrow H^0(C, N),$$

where $N = \mathcal{O}_C(C)$ is the normal bundle to C in X.

LEMMA 1.1. *Assume that $m_{x_t}(C_t) \geq m$ for all $t \in \Delta$. Then $\rho(\frac{d}{dt}) \in H^0(C, N)$ vanishes to order $\geq (m-1)$ at x.*

REMARK. We say that a section $s \in H^0(C, N)$ vanishes to order $\geq k$ at a (possibly singular) point $y \in C$ if s is actually a section of the subsheaf $N \otimes \mathfrak{m}_y^k \subset N$, where $\mathfrak{m}_y$ is the maximal ideal sheaf of y.

PROOF OF LEMMA 1.1: We simply make an explicit computation. Specifically, the assertion is local on C and Δ, so we can assume that Δ is a small disk with coordinate t, and that C lies in an open subset U of $\mathbb{C}^2$ with coordinates (z, w), and $x = (0, 0)$. The total space $\mathcal{C} \subset U \times \Delta$ of the deformation is then defined by a power series $F(z, w, t) = f_t(z, w)$ where $C_t = \{f_t = 0\}$. We may suppose that $x_t = (a(t), b(t))$ for suitable power series $a(t), b(t)$. Then the curve defined by

$$\phi_t(z, w) =_{\text{def}} F(z + a(t), w + b(t), t)$$

has multiplicity $\geq m$ at $(0, 0)$ for all $t \in \Delta$. Expanding $\phi_t(z, w) = \sum \phi_i(z, w) t^i$ as a power series in t, it follows that $\phi_i \in (z, w)^m$ for all i. On the other hand,

$$\phi_1(z, w) = \frac{\partial f_0}{\partial z}(z, w) \cdot a'(0) + \frac{\partial f_0}{\partial w}(z, w) \cdot b'(0) + \frac{\partial F}{\partial t}(z, w, 0),$$

and since $\frac{\partial f_0}{\partial z}(z, w), \frac{\partial f_0}{\partial w}(z, w) \in (z, w)^{m-1}$, we find that

$$\frac{\partial F}{\partial t}(z, w, 0) \in (z, w)^{m-1}.$$

But $\frac{\partial F}{\partial t}|C$ is the local expression for $\rho(\frac{d}{dt}) \in H^0(C, N)$, and the lemma follows. ∎

COROLLARY 1.2. *In the situation of the Lemma, assume in addition that C is reduced and irreducible, and that the Kodaira-Spencer deformation class $\rho(\frac{d}{dt}) \in H^0(C,N)$ is non-zero. Then $C \cdot C \geq m(m-1)$.*

PROOF: This follows from the Lemma plus the fact that $c_1(N)$ represents $C \cdot C$. In more detail, let $f : Y \longrightarrow X$ be the blowing-up of X at x, with exceptional divisor $E \subset Y$. Then $f^*C = C' + kE$, where $C' \subset Y$ is the proper transform of C, and $k = m_x(C) \geq m$. Note that C' is the blowing-up of C at x. Put $s = \rho(\frac{d}{dt})$, so that $0 \neq s \in H^0(C, \mathfrak{m}_x^{m-1} \otimes \mathcal{O}_C(C))$. Then s induces a non-zero section

$$s' \in H^0(C', f^*(\mathcal{O}_C(C)) \otimes \mathcal{O}_Y((1-m)E)|_{C'}).$$

This implies that $\deg f^*(\mathcal{O}_C(C))|_{C'} \geq (m-1)E \cdot C' = k(m-1)$. It follows that

$$C \cdot C = \deg \mathcal{O}_C(C) = \deg f^*(\mathcal{O}_C(C))|_{C'} \geq k(m-1) \geq m(m-1),$$

as claimed. ∎

§2. Proof of the Theorem

We now give the proof of the theorem stated in the Introduction.

As in the statement, let L be an ample line bundle on the smooth surface X. Then there are only finitely many algebraic families of reduced irreducible (i.e. integral) curves on X of bounded degree with respect to L. Therefore for fixed $d > 0$ the set

$$S_d = \left\{ (C,x) \;\middle|\; x \in C \subset X \text{ an integral curve }, \; m_x(C) > C \cdot L, \; C \cdot L \leq d \right\}$$

is parametrized by a finite union of irreducible quasi-projective varieties. Consequently

$$S = \left\{ (C,x) \;\middle|\; x \in C \subset X \text{ a reduced irreducible curve }, \; m_x(C) > C \cdot L \right\}$$

consists of at most countably many algebraic families. The first statement of the theorem will follow if we prove that each of these families is discrete.

Suppose to the contrary that there exists a non-trivial continuous family $\{ (C_t, x_t) \}_{t \in \Delta}$ of reduced irreducible curves $C_t \subset X$, plus points $x_t \in C_t$, with

$$(*) \qquad m_t =_{\text{def}} mult_{x_t}(C_t) > C_t \cdot L \qquad \text{for all } t \in \Delta.$$