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Introduction

The complicated nature of the theory of cycles of codimension two and higher became apparent with Mumford's paper [M], which showed that $p_g = 0$ is a necessary condition for the representability of the group of zero-cycles on a smooth projective surface over $\mathbb{C}$. This was generalized by Roitman [R] when he showed that the vanishing of all the groups $H^0(\Omega^q)$, $q > 1$, is necessary for the representability of the group of zero-cycles on a smooth projective variety over $\mathbb{C}$. Bloch, Kas and Lieberman [BKL] investigated the zero-cycles on surfaces with $p_g = 0$, showing that the group of zero- cycles was in fact representable, at least if the surface is not of general type; Bloch [Bl] has conjectured that $p_g = 0$ is sufficient for the representability of the zero-cycles on a smooth projective surface. The case of surfaces of general type is still an open problem, although there has been some progress, most recently by Voisin [V].

Bloch's proof in [Bl] of Mumford's infinite dimensionality theorem views the diagonal in $X \times X$ as a family of zero-cycles on X, parametrized by X, and goes on to consider the consequences of the generic triviality of this family. This may be the first appearance of this point of view. Coombes and Srinivas used this idea in [CS] to get a decomposability result for $H^1(\mathcal{K}_2)$ of a surface. Bloch and Srinivas [BS] push this approach further, making a study of the cycle groups on a smooth variety X which relies on a partial decomposition of the diagonal in $X \times X$. They have applied this method to give some examples for which certain cycle groups are representable. This approach was recently used by Paranjape [P] in his discussion of the cycle groups of subvarieties of projective space of small degree and small codimension. Schoen [S] has also applied this method to give generalizations of the Mumford-Roitman criterion for non-representability to the Chow groups of cycles of positive dimension. Jannsen [J] used the ideas of Bloch and Srinivas in his discussion of smooth projective varieties X for which the rational topological cycle maps

$$\mathrm{CH}^p(X) \otimes \mathbb{Q} \to H^{2p}_{\mathcal{B}}(X, \mathbb{Q})$$

are injective. For such a variety, Jannsen shows that the diagonal in $X \times X$ decomposes in $\mathrm{CH}^*(X \times X)_{\mathbb{Q}}$ into a sum of product cycles

$$\Delta = A_0 \times B^0 + A_1 \times B^1 + \ldots + A_d \times B^d$$

where A_i is a dimension i cycle, B^i is a codimension i cycle, and $d = \dim(X)$. One consequence of this decomposition is that the total cycle map

$$\bigoplus_{p=0}^{d} \mathrm{CH}^p(X) \otimes \mathbb{Q} \to \bigoplus_{q=0}^{2d} H^q_B(X, \mathbb{Q})$$

is an isomorphism; in particular, X has no odd cohomology.

In this paper, we prove an analog of Jannsen's result, considering the cycle map to rational Deligne cohomology rather than Betti cohomology. Assuming injectivity of the Deligne cycle maps, we arrive at a decomposition of the diagonal into a sum of codimension one cycles on products of the form $\Gamma_{i+1} \times D^i$, with $\dim(\Gamma_{i+1}) = i+1$, $\mathrm{cod}(D^i) = i$ (see Theorem 1.2 for a more precise statement). The consequences of this decomposition are a surjectivity statement for certain cycle maps to Deligne cohomology and some other related maps (Theorem 2.5), a vanishing result for certain Hodge numbers (Theorem 3.2), and a decomposability result for the K-cohomology (Theorem 4.1). If we assume that *all* the rational cycle class maps for a smooth projective variety X are injective, then

(1) all the rational Hodge cycles on X are algebraic (Corollary 2.6)

(2) the Abel-Jacobi maps

$$cl^n \colon \mathrm{CH}^n(X)_{alg} \to J^n(X)$$

are all surjective (Corollary 3.3)

(3) the Hodge numbers $h^{p,q}(X)$ all vanish for $|p - q| > 1$.

(4) the maps

$$\mathrm{CH}^p(X) \otimes \mathbb{C}^\times \to H^p(X, \mathcal{K}_{p+1})$$

are all surjective.

The results on the Hodge numbers are a direct generalization of the results of Mumford-Roitman mentioned above. This points the way to some possible generalizations of Bloch's conjecture to a conjecture on the representability of cycle groups of higher dimension (see Questions 1 and 2 in §3). What is novel about the situation is that it involves *all* the groups of cycles of dimension 0 to s rather than the cycles of a single dimension s. Schoen has raised similar questions in his paper [S], from a slightly different point of view, replacing the injectivity assumption with an assumption that the generalized Hodge conjecture holds, and that the group of dimension s cycles is representable; we haven't attempted to reconcile these two points of view.

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§1. Decomposition of the diagonal

In this section, we show how the injectivity of the cycle map to Deligne cohomology leads to a decomposition of the diagonal. If X is a smooth projective variety, we let $\mathcal{Z}^n(X)$ denote the group of codimension n cycles on X, $\mathrm{CH}^n(X)$ the group of cycles modulo rational equivalence. We let $\mathcal{Z}_n(X)$ and $\mathrm{CH}_n(X)$ denote the group of dimension n cycles and cycle classes. If X is defined over $\mathbb{C}$, we have the cycle class map

$$cl^n\colon \mathcal{Z}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

This map passes to rational equivalence, giving the map

$$cl^n\colon \mathrm{CH}^n(X) \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)).$$

We refer to an element of $\mathcal{Z}^n(X)_{\mathbb{Q}}$ as a $\mathbb{Q}$-cycle. We also denote by cl^n the maps induced by cl^n after extending the coefficient ring. For the basic properties of Deligne cohomology and the cycle map, we refer the reader to [B].

Let $Hg^n(X)$ denote the group of codimension n *Hodge cycles* on X:

$$Hg^n(X) := \{x \in H^{2n}(X, \mathbb{Z}(n)) \mid x \otimes 1 \in F^n H^{2n}(X, \mathbb{C})\}.$$

We have the exact sequence describing $H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n))$ as an extension:

$$0 \to \frac{H^{2n-1}(X, \mathbb{C})}{H^{2n-1}(X, \mathbb{Z}(n)) + F^n H^{2n-1}(X, \mathbb{C})} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Z}(n)) \to Hg^n(X) \to 0.$$

The n^{th} intermediate Jacobian, $J^n(X)$, is the complex torus on the left-hand side of the above sequence.

Lemma 1.1. *Let X be a smooth projective variety over $\mathbb{C}$ of dimension d. Suppose the $\mathbb{Q}$-cycle class map*

$$cl^n \colon CH^n(X)_{\mathbb{Q}} \to H^{2n}_{\mathcal{D}}(X, \mathbb{Q}(n))$$

is injective. Let D be a pure codimension $i = d - n$ closed subset of X, and let γ be a codimension d $\mathbb{Q}$-cycle on $X \times X$, supported on $X \times D$. Then there are closed subsets D' and Γ of X, codimension d $\mathbb{Q}$-cycles $\gamma_?$ and $\gamma^?$ on $X \times X$ such that

(1) D' has pure codimension $i + 1$ and Γ has pure dimension $i + 1$.

(2) $\gamma_?$ is supported on $\Gamma \times D$ and $\gamma^?$ is supported on $X \times D'$.

(3) $\gamma = \gamma_? + \gamma^?$ in $CH^d(X \times X)_{\mathbb{Q}}$.

Proof. If D has irreducible components $D_1, \ldots, D_s$, we can write γ as a sum

$$\gamma = \gamma^1 + \ldots + \gamma^s$$

with γ^j supported on $X \times D_j$. Thus we may assume that D is irreducible. Write γ as a sum, $\gamma = \gamma' + \gamma''$, such that each irreducible component of the support of γ' dominates D, and no irreducible component of the support of γ'' dominates D. Since γ'' is supported on $X \times p_2(supp(\gamma''))$, and $p_2(supp(\gamma''))$ has codimension at least $i+1$ on X, we may assume that $\gamma = \gamma'$. We may then