Astérisque

HÉLÈNE ESNAULT
V. SRINIVAS
ECKART VIEHWEG

Decomposability of Chow groups implies decomposability of cohomology

Astérisque, tome 218 (1993), p. 227-241
<http://www.numdam.org/item?id=AST_1993__218__227_0>

Decomposability of Chow groups implies decomposability of cohomology

Hélène Esnault, V. Srinivas and Eckart Viehweg

Let X be an n-dimensional complete irreducible smooth variety defined over the field $\mathbb{C}$ of complex numbers. For any Zariski open subset V of X, we have the following graded rings.

(i) $\bigoplus_{i=0}^{n} CH^i(V)_{\mathbb{Q}}$, where $CH^i(V)_{\mathbb{Q}}$ is the Chow group of algebraic cycles of codimension i on V with rational coefficients, modulo rational equivalence (see [F], Chapter 8, Prop. 8.3).

(ii) $\bigoplus_{i=0}^{n} H^i(V)/N^1H^i(V)$, where $H^i(V) = H^i(V_{an}, \mathbb{Q})$ is the singular cohomology of the underlying complex manifold V_{an}, and

$$N^aH^i(V) = \varinjlim_{\operatorname{codim} Z \geq a} \ker\Big(H^i(V) \longrightarrow H^i(V - Z)\Big)$$

defines Grothendieck's coniveau filtration (here Z runs over the Zariski closed subsets of V of codimension $\geq a$).

(iii) $\bigoplus_{i=0}^{n} H^0(V, \mathcal{H}_V^i)$, where $\mathcal{H}_V^i$ is the sheaf for the Zariski topology associated to the presheaf

$$U \longmapsto H^i(U) = H^i(U_{an}, \mathbb{Q}).$$

(iv) We also have a graded ring associated to X: $\bigoplus_{i=0}^{n} H^i(\mathbb{C}(X))$, where

$$\begin{aligned} H^i(\mathbb{C}(X)) &:= \varinjlim_{V \subset X} H^i(V) = \varinjlim_{V \subset X} H^i(V)/N^1H^i(V) \\ &= \varinjlim_{V \subset X} H^0(V, \mathcal{H}_V^i) \end{aligned}$$

Here the direct limits are over the non-empty Zariski open sets V in X, and $\mathbb{C}(X)$ denotes the function field of X. The first equality defines the cohomology of the function field; the right side of the equality is clearly a birational invariant of X.

In (ii), (iii), (iv) above, we consider only cohomology in degrees up to n, since the singular cohomology of an affine variety of dimension n vanishes in degrees larger than n, by the weak Lefschetz theorem (this implies that for any variety V of dimension n, we have $H^i(V) = N^1H^i(V)$ for $i > n$).

Theorem 1 *Let X be a smooth complete variety of dimension n over $\mathbb{C}$. Suppose there exists a non empty Zariski open subset $V \subset X$, and positive integers $n_1, \dots, n_r$ with $\sum_i n_i = n$, such that one of the following product maps is surjective:*

(i) $CH^{n_1}(V)_{\mathbb{Q}} \otimes \cdots \otimes CH^{n_r}(V)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$

(ii) $H^{n_1}(V)/N^1H^{n_1}(V) \otimes \cdots \otimes H^{n_r}(V)/N^1H^{n_r}(V) \longrightarrow H^n(V)/N^1H^n(V)$

(iii) $H^0(V, \mathcal{H}_V^{n_1}) \otimes \cdots \otimes H^0(V, \mathcal{H}_V^{n_r}) \longrightarrow H^0(V, \mathcal{H}_V^n)$

(iv) $H^{n_1}(\mathbb{C}(X)) \otimes \cdots \otimes H^{n_r}(\mathbb{C}(X)) \longrightarrow H^n(\mathbb{C}(X))$

Then the cup product map for the coherent cohomology

$$H^{n_1}(X, \mathcal{O}_X) \otimes H^{n_2}(X, \mathcal{O}_X) \otimes \cdots \otimes H^{n_r}(X, \mathcal{O}_X) \longrightarrow H^n(X, \mathcal{O}_X) \qquad (*)$$

is surjective.

The proof of (i) is motivated by Bloch's proof [B] of Mumford's theorem that for surfaces X with $H^2(X, \mathcal{O}_X) \neq 0$, the Chow group of 0-cycles $CH^2(X)$ is not 'finite dimensional' (see also the 'metaconjecture' in Chapter 1 of [B2]). Many other variants of Bloch's method have been considered by several authors. The method involves the action of correspondences on the cohomology. At the referee's suggestion, we try to make this argument with some care, though this type of reasoning is well known to experts.

The proofs of (ii), (iii) and (iv) are a consequence of the mixed Hodge structure on the cohomology of the open sets V (see [D]). For $V = X$, the surjectivity of the map (ii) trivially implies that $(*)$ is surjective, using the Hodge decomposition on cohomology, since the ring $\oplus H^i(X, \mathcal{O}_X)$ is a graded quotient of $\oplus(H^i(X)/N^1H^i(X)) \otimes \mathbb{C}$.

The proof of the theorem

We first discuss (i). Let $C = X - V$, and let $k \subset \mathbb{C}$ be a countable algebraically closed field of definition of X, C and V. Let X_0, C_0, V_0 be the corresponding models over k, and for any extension L of k, let $X_L = X_0 \times_k L$, etc. We embed $k(X_0) \hookrightarrow \mathbb{C}$ as a k-subalgebra, and consider the generic point of X_0 as a closed point $\eta \in X_{k(X_0)}$, hence as an element of $CH^n(X_{k(X_0)})_{\mathbb{Q}}$. By assumption, its image under the composite

$$CH^n(X_{k(X_0)})_{\mathbb{Q}} \longrightarrow CH^n(X)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

decomposes as

$$\sum_{\text{finite}} m_{n_1} \cdot \cdots \cdot m_{n_r}$$

where $m_{n_i} \in CH^{n_i}(V)_{\mathbb{Q}}$. The m_{n_i} are defined over a subfield $L \subset \mathbb{C}$ which is finitely generated over $k(X_0)$, and (see [B2], Lecture 1, Appendix, Lemma 3) the natural map

$$CH^n(V_L)_{\mathbb{Q}} \longrightarrow CH^n(V)_{\mathbb{Q}}$$

is injective, so

$$\sum_{\text{finite}} m_{n_1} \cdot \cdots \cdot m_{n_r} = [\eta] \tag{1}$$

holds in $CH^n(V_L)_{\mathbb{Q}}$.

Let F be the algebraic closure of $k(X_0)$ in L; since L is finitely generated over $k(X_0)$, F is a finite algebraic extension of $k(X_0)$. We can find a non-singular affine F-variety W with function field L. The graded ring

$$\bigoplus_{i \geq 0} CH^i(V_L)$$

is the direct limit of the graded rings

$$\bigoplus_{i \geq 0} CH^i(V_F \times_F W'),$$

where W' runs over the non-empty Zariski open sets in W (see [B2], Lecture 1, Appendix, Lemma 1). So after replacing W by a nonempty open subset, we may assume given classes $m_{n_i} \in CH^{n_i}(V_F \times_F W)$ such that (1) holds in

$$CH^n(V_F \times_F W)_{\mathbb{Q}},$$

where $[\eta]$ now denotes the image in $CH^n(V_F \times_F W)_{\mathbb{Q}}$ of the earlier class

$$[\eta] \in CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_F)_{\mathbb{Q}}.$$

Let $P \in W$ be a closed point. Then there is a homomorphism of rings

$$f^* : \bigoplus_{i \geq 0} CH^i(V_F \times_F W) \to \bigoplus_{i \geq 0} CH^i(V_F \times_F \operatorname{Spec} F(P)),$$

where $f : V_F \times_F \operatorname{Spec} F(P) \to V_F \times_F W$ is induced by the inclusion of P into W (f is a morphism of non-singular F-varieties, hence by [F], Prop. 8.3, such a homomorphism f^* exists). Then $f^*[\eta]$ is just $[\eta]$ considered as an element of $CH^n(V_{k(X_0)})_{\mathbb{Q}} \subset CH^n(V_{F(P)})_{\mathbb{Q}}$. Hence

$$\sum_{\text{finite}} f^*(m_{n_1}) \cdot \dots \cdot f^*(m_{n_r}) = [\eta] \tag{2}$$

holds in $CH^n(V_{F(P)})_{\mathbb{Q}}$, where $f^*(m_{n_i}) \in CH^{n_i}(V_{F(P)})_{\mathbb{Q}}$.

Hence, we are reduced to the situation when (1) holds, where L is a finite algebraic extension of $k(X_0)$, and $m_{n_i} \in CH^i(V_L)_{\mathbb{Q}}$.

By resolution of singularities, we can find a projective non-singular k-variety Z_0, together with a k-morphism $\sigma_0 : Z_0 \to X_0$, such that the induced map on function fields is the given inclusion $k(X_0) \to L$. Since L is a finite extension of $k(X_0)$, the morphism σ_0 is generically finite.

The (flat) k-morphism $\operatorname{Spec} L \to Z_0$ given by the inclusion of the generic point gives rise to a natural surjective homomorphism of graded rings

$$Cl : \bigoplus_{i \geq 0} CH^i(X_0 \times_k Z_0)_{\mathbb{Q}} \to \bigoplus_{i \geq 0} CH^n(V_L)_{\mathbb{Q}},$$

such that if $[\Delta_{\sigma_0}] \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}$ is the class of the transposed graph of σ_0, then $Cl([\Delta_{\sigma_0}])$ is just $[\eta] \in CH^n(V_L)_{\mathbb{Q}}$. The kernel of

$$CH^n(X_0 \times_k Z_0)_{\mathbb{Q}} \to CH^n(V_L)$$

consists of the subgroup generated by the classes supported on subsets of the form $(C_0 \times_k Z_0) \cup (X_0 \times_k D_0)$, as D_0 runs over all proper subvarieties of Z_0 (see [B2], Lecture 1, Appendix, Lemma 1, and [F], Prop. 1.8). Thus we have an equation

$$[\Delta_{\sigma_0}] - \sum M_{n_1} \cdot \dots \cdot M_{n_r} = \gamma_0 + \delta_0$$

in $CH^n(X_0 \times_k Z_0)$, where for some divisor $D_0 \subset Z_0$, we have

$$\begin{array}{rlrl} M_{n_i} \in CH^{n_i}(X_0 \times_k Z_0)_{\mathbb{Q}}, & & M_{n_i} \mapsto & m_{n_i} \in CH^{n_i}(V_L)_{\mathbb{Q}} \\ \gamma_0 \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & & \operatorname{supp} \gamma_0 \subset & C_0 \times_k Z_0 \\ \delta_0 \in CH^n(X_0 \times_k Z_0)_{\mathbb{Q}}, & & \operatorname{supp} \delta_0 \subset & X_0 \times_k D_0 \end{array}$$