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The study of moduli spaces by means of the period mapping has found its greatest success for moduli spaces of varieties with trivial canonical bundle, or more generally, varieties with Kodaira dimension zero. Now these moduli spaces play a pivotal rôle in the classification theory of algebraic varieties, since varieties with nonnegative Kodaira dimension which are not of general type admit birational fibrations by varieties of Kodaira dimension zero. Since such fibrations typically include singular fibers as well as smooth ones, it is important to understand how to compactify the corresponding moduli spaces (and if possible, to give geometric interpretations to the boundary of the compactification). Note that because of the possibility of blowing up along the boundary, abstract compactifications of moduli spaces are far from unique.

The hope that the period mapping could be used to construct compactifications of moduli spaces was given concrete expression in some conjectures of Griffiths [25, §9] and others in the late 1960's. In particular, Griffiths conjectured that there would be an analogue of the Satake-Baily-Borel compactifications of arithmetic quotients of bounded symmetric domains, with some kind of "minimality" property among compactifications. Although there has been much progress since [25] in understanding the behavior of period mappings near the boundary of moduli, compactifications of this type have not been constructed, other than in special cases.

In the case of algebraic K3 surfaces, the moduli spaces themselves are arithmetic quotients of bounded symmetric domains, so each has a minimal (Satake-Baily-Borel) compactification. In studying the moduli spaces for K3 surfaces of low degree in the early 1980's, Looijenga [35] found that the Satake-Baily-Borel compactification needed to be blown up slightly in order to give a good geometric interpretation to the boundary. He introduced a class of compactifications, the *semi-toric compactifications*, which includes the ones with a good geometric interpretation.

In higher dimension, the moduli spaces are not expected to be arithmetic quotients of symmetric domains, so different techniques are needed. The study of these moduli spaces has received renewed attention recently, due to the discovery by theoretical physicists of a phenomenon called "mirror symmetry". One of the predictions of mirror symmetry is that the moduli space for a variety with trivial canonical bundle, which parameterizes the possible complex structures on the underlying differentiable manifold, should *also* serve as the parameter space for a very different kind of structure on a "mirror partner"—another variety with trivial canonical bundle. This alternate description of the moduli space turns out to be well-adapted to analysis by Looijenga's techniques; we carry out that analysis here.

In the physicists' formulation, one fixes a differentiable manifold X which admits complex structures with trivial canonical bundle (a "Calabi-Yau manifold"), and studies something called nonlinear sigma-models on X. Such an object can be determined by specifying both a complex structure on X, and some "extra structure" (cf. [40]); the moduli space of interest to the physicists parameterizes the choice of both. The rôles of the "complex structure" and "extra structure" subspaces of this parameter space are reversed when X is replaced by a mirror partner.

Most aspects of mirror symmetry must be regarded as conjectural by mathematicians at the moment, and in this paper we conjecture much more than we prove. In a companion paper [41], we consider formally degenerating variations of Hodge structure near normal crossing boundary points of the moduli space, and describe a conjectural link to the numbers of rational curves of various degrees on a mirror partner. In the present paper, we extend these considerations to boundary points which are *not* of normal crossing type, and formulate a mathematical mirror symmetry conjecture in greater generality. In addition, we find that when studied from the mirror perspective, a "minimal" partial compactification of the moduli space—analogous to the Satake-Baily-Borel compactification—appears very natural, provided that several conjectures about the mirror partner hold.

One of our conjectures is a simple and compelling statement about the Kähler cone of Calabi-Yau varieties. If true, it clarifies the rôle of some of the "infinite discrete" structures on such a variety, which nevertheless seem to be finite modulo automorphisms. We have verified this conjecture in a nontrivial case in joint work with A. Grassi [21].

The plan of the paper is as follows. In the first several sections, we review Looijenga's compactifications, describe a concrete example, and add a refinement to the theory in the form of a flat connection on the holomorphic cotangent bundle of the moduli space. We then turn to the description of the larger moduli spaces of interest to physicists, and analyze certain boundary points of those spaces. Towards the end of the paper, we explore the mathematical implications of mirror symmetry in constructing compactifications of moduli spaces. We close by discussing some evidence for mirror symmetry which (in hindsight) was available in 1979.

1 Semi-toric compactifications

The first methods for compactifying arithmetic quotients of bounded symmetric domains were found by Satake [46] and Baily-Borel [5]. The compactification produced by their methods, often called the *Satake-Baily-Borel compactification*, adds a "minimal" amount to the quotient space in completing it to a compact complex analytic space. This minimality can be made quite precise, thanks to the Borel extension theorem [10] which guarantees that for a given quotient of a bounded symmetric domain by an arithmetic group, any compactification whose boundary is a divisor with normal crossings will map to the Satake-Baily-Borel compactification (provided that the arithmetic group is torsion-free).

Satake-Baily-Borel compactifications have rather bad singularities on their boundaries, so they are difficult to study in detail. Explicit resolutions of singularities for these compactifications were constructed in special cases by Igusa [30], Hemperly [27], and Hirzebruch [28]; the general case was subsequently treated by Satake [47] and Ash et al. [1]. The methods of [1] produce what are usually called *Mumford compactifications*—these are smooth, and have a divisor with normal crossings on the boundary, but unfortunately many choices must be made in their construction. The Satake-Baily-Borel compactification, on the other hand, is canonical.

Some years later, Looijenga [35] generalized both the Satake-Baily-Borel and the Mumford compactifications by means of a construction which can be applied widely, not just in the case of arithmetic quotients of bounded symmetric domains. Looijenga's construction gives partial compactifications of certain quotients of tube domains by discrete group actions. A *tube domain* is the set of points in a complex vector space whose imaginary parts are constrained to lie in a specified cone. Whereas Ash et al. [1] had only considered homogeneous self-adjoint cones, Looijenga showed that analogous constructions could be made in a more general context.

The starting point is a free $\mathbb{Z}$-module L of finite rank, and the real vector space $L_{\mathbb{R}} := L \otimes \mathbb{R}$ which it spans. A convex cone σ in $L_{\mathbb{R}}$ is *strongly convex* if $\sigma \cap (-\sigma) \subset \{0\}$. A convex cone is *generated* by the set S if every element in the cone can be written as a linear combination of the elements of S with nonnegative coefficients. And a convex cone is *rational polyhedral* if it is generated by a finite subset of the rational vector space $L_{\mathbb{Q}} := L \otimes \mathbb{Q}$.

Let $\mathcal{C} \subset L_{\mathbb{R}}$ be an open strongly convex cone, and let $\Gamma \subset \mathrm{Aff}(L)$ be a group of affine-linear transformations of L which contains the translation subgroup L of $\mathrm{Aff}(L)$. If the linear part $\Gamma_0 := \Gamma/L \subset \mathrm{GL}(L)$ of Γ preserves the cone $\mathcal{C}$, then the group Γ acts on the tube domain $\mathcal{D} := L_{\mathbb{R}} + i\,\mathcal{C}$. We wish to partially compactify the quotient space $\mathcal{D}/\Gamma$, including limit points for all paths moving out towards infinity in the tube domain.

Looijenga formulated a condition which guarantees the existence of partial compactifications of this kind. Let $\mathcal{C}_+$ be the convex hull of $\overline{\mathcal{C}} \cap L_{\mathbb{Q}}$. Following [35], we say that $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$ is *admissible* if there exists a rational polyhedral cone $\Pi \subset \mathcal{C}_+$ such that $\Gamma_0.\Pi = \mathcal{C}_+$. Given an admissible triple $(L_{\mathbb{Q}}, \mathcal{C}, \Gamma_0)$, the (somewhat cumbersome) data needed to specify one of Looijenga's partial compactifications is as follows.[1]

DEFINITION 1 [35] *A* locally rational polyhedral decomposition *of $\mathcal{C}_+$ is a collection $\mathcal{P}$ of strongly convex cones such that*

(i) $\mathcal{C}_+$ is the disjoint union of the cones belonging to $\mathcal{P}$,

(ii) for every $\sigma \in \mathcal{P}$, the $\mathbb{R}$-span of σ is defined over $\mathbb{Q}$,

(iii) if $\sigma \in \mathcal{P}$, if τ is the relative interior of a nonempty face of the closure of σ, and if $\tau \subset \mathcal{C}_+$, then $\tau \in \mathcal{P}$, and

(iv) if Π is a rational polyhedral cone in $\mathcal{C}_+$, then Π meets only finitely many members of $\mathcal{P}$.

(The decomposition $\mathcal{P}$ is called *rational polyhedral* if all the cones in $\mathcal{P}$ are relative interiors of rational polyhedral cones. This is the same notion which appears in toric geometry [19, 43], except that the cones appearing in $\mathcal{P}$ as formulated here are the relative interiors of the cones appearing in that theory.)

For each Γ_0-invariant locally rational polyhedral decomposition $\mathcal{P}$ of $\mathcal{C}_+$, there is a partial compactification of $\mathcal{D}/\Gamma$ called the *semi-toric (partial) compactification associated to $\mathcal{P}$*. This partial compactification has the form $\widehat{\mathcal{D}}(\mathcal{P})/\Gamma$, where $\widehat{\mathcal{D}}(\mathcal{P})$ is the disjoint union of certain strata $\mathcal{D}(\sigma)$ associated to the cones σ in the decomposition. The complex dimension of the stratum $\mathcal{D}(\sigma)$ coincides with the real codimension of the cone σ in $L_{\mathbb{R}}$; in particular, the open cones in $\mathcal{P}$ correspond to the 0-dimensional strata in $\widehat{\mathcal{D}}(\mathcal{P})$. The delicate points in the construction are the specification of a topology on $\widehat{\mathcal{D}}(\mathcal{P})$,

[1] We have modified Looijenga's definition slightly, so that the use of the term "face" is the standard one (cf. [45]): a subset $\mathcal{F}$ of a convex set $\mathcal{S}$ is a *face* of $\mathcal{S}$ if every closed line segment in $\mathcal{S}$ which has one of its relative interior points lying in $\mathcal{F}$ also has both endpoints lying in $\mathcal{F}$.