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ALEXANDRU BUIUM

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# A Finiteness Theorem for Isogeny Correspondences

**Alexandru Buium**

## 0. Introduction

Let $A_{g,n}$ be the moduli space of principally polarized abelian varieties over $\mathbb{C}$ of dimension $g \geq 2$ with level $n$ structure, $n \geq 3$; we will view $A_{g,n}$ as an algebraic variety over $\mathbb{C}$. Moreover, let $Y \subset A_{g,n}$ be a curve (by which we will understand an irreducible, closed, possibly singular subvariety of dimension 1). By an isogeny correspondence on $Y$ we will understand an (irreducible, closed, possibly singular) curve $Z \subset Y \times Y$ for which there exists a quasi-finite map $Z' \to Z$ from an irreducible curve $Z'$ with the property that the two abelian schemes over $Z'$ deduced by base change via

$$Z' \to Z \subset Y \times Y \xrightarrow{p_i} Y \qquad i = 1, 2$$

($p_i$ = i-th projection) are isogenous. Note that two abelian schemes over $Z'$ are called isogenous if there exists a surjective homomorphism between them with kernel finite over $Z'$; so we do not require our isogenies preserve, say, polarizations.

The question which we address in this paper is: how many isogeny correspondences can exist on a "sufficiently general" curve $Y \subset A_{g,n}$?

It is easy to see that there exist "lots" of curves $Y \subset A_{g,n}$ carrying infinitely many isogeny correspondences: more precisely, the union of all such

$Y$'s in $A_{g,n}(\mathbb{C})$ is dense in the complex topology of $A_{g,n}(\mathbb{C})$ (see the Proposition from Section 1). Nevertheless, our main result here will imply in particular that "most" curves $Y \subset A_{g,n}$ carry at most finitely many isogeny correspondences (see Theorem 1 below).

Indeed, let $C(A_{g,n})$ be the set of all (irreducible, closed, possibly singular) curves in $A_{g,n}$; we will put a natural topology on $C(A_{g,n})$ which we call the Kolchin topology such that $C(A_{g,n})$ becomes an irreducible Noetherian topological space and then we will prove in particular the following:

**Theorem 1.** *There exists a dense Kolchin open subset $C_0$ of $C(A_{g,n})$ such that any curve $Y$ belonging to $C_0$ carries at most finitely many isogeny correspondences.*

**Remark.** If a curve $Y \subset A_{g,n}$ carries at most finitely many isogeny correspondences $Z$ then any such $Z$ must have only finite orbits.

Let's define in what follows the Kolchin topology on $C(A_{g,n})$. More generally one can define the Kolchin topology on the set $C(A)$ of all (irreducible, closed, possibly singular) curves embedded in a given (irreducible, possibly singular) algebraic variety $A$ over $\mathbb{C}$. Indeed, we consider first the "jet scheme" $\mathrm{jet}\,(A)$, cf. [$B_1$]; recall that this is by definition an $A$-scheme with a $\mathbb{C}$-derivation $\delta$ of its structure sheaf, characterized by the fact that for any pair $(Z, d)$ consisting of an $A$-scheme $Z$ and a $\mathbb{C}$-derivation $d$ on $\mathcal{O}_Z$ there is a unique horizontal morphism of $A$-schemes $Z \to \mathrm{jet}\,(A)$; "horizontal" here means "commuting with $\delta$ and $d$". For instance, if $A = \mathbb{A}^n = \mathrm{Spec}\,\mathbb{C}[y_1, \dots, y_n]$ then $\mathrm{jet}\,(A) = \mathrm{Spec}\,\mathbb{C}\{y_1, \dots, h_n\}$ where $\mathbb{C}\{y_1, \dots, y_n\}$ is the ring of $\delta$-polynomials in $y_1, \dots, y_n$ with coefficients in $\mathbb{C}$ (which by definition is the ring of polynomials with coefficients in $\mathbb{C}$ in the infinite family of variables $y_j^{(i)}$, $i \geq 0$, $1 \leq j \leq n$, with $\mathbb{C}$-derivation $\delta$ sending $y_j^{(i)}$ into $y_j^{(i+1)}$). Now for any Zariski closed subset $H$ of $\mathrm{jet}\,(A)$ we denote by $C_H(A)$ the set of all curves $Y \in C(A)$ such that the image of the natural horizontal closed immersion $\mathrm{jet}\,(Y) \to \mathrm{jet}\,(A)$ is contained in $H$. One easily checks that the sets $C_H(A)$ are the closed sets of a topology which we call the Kolchin topology (one has to use the non-obvious fact that $\mathrm{jet}\,(Y)$ is an irreducible scheme which follows from correctly interpreting a theorem of Kolchin, [K] p. 200). We will check in Section 2 below that $C(A)$ with the Kolchin topology is an irreducible Noetherian topological space.

**Remark.** Intuitively a subset of $C(A)$ is Kolchin closed if it consists of all curves $Y \in C(A)$ which "satisfy a certain system of algebraic differential equations on $A$". As the proof of Theorem 1 will show, the "system defining" $C(A_{g,n}) \setminus C_0$ has "order 6" (i.e. "comes from jets of order 6") and is highly nonlinear.

Actually we can do much better than in Theorem 1, namely we can "bound asymptotically" (for $Y \in C_0$) the number of isogeny correspondences on $Y$ "counted with certain natural multiplicities" (see Theorem 1′ below). We need more notations. For any curve $Y \subset A_{g,n}$ we denote by $p(Y)$ the genus of a smooth projective model of $Y$. Moreover, for any isogeny correspondence $Z \subset Y \times Y$ we let $[Z : Y]_i$ denote the degree of the map $Z \subset Y \times Y \xrightarrow{p_i} Y$, $i = 1, 2$ and put $i(Y) = \sum [Z : Y]_1 = \sum [Z : Y]_2 \in \mathbb{N} \cup \{\infty\}$, where $Z$ runs through the set of all isogeny correspondences on $Y$ (we put $i(Y) = 0$ if this set is empty). This $i(Y)$ is the "number of isogeny correspondences counted with multiplicities"; for alternative descriptions of $i(Y)$ we refer to Lemmas 1 and 2 from Section 1. Finally, we shall fix a smooth projective compactification

$\bar{A}_{g,n}$ of $A_{g,n}$ and a very ample line bundle $\mathcal{O}(1)$ on $\bar{A}_{g,n}$; then for any curve $Y \subset A_{g,n}$ we shall denote by $\deg(Y)$ the degree of the Zariski closure of $Y$ in $\bar{A}_{g,n}$ with respect to $\mathcal{O}(1)$.

We can state the following strengthening of Theorem 1:

**Theorem 1′.** *There exist a dense Kolchin open subset $C_0$ of $C(A_{g,n})$ and two positive integers $m_1$, $m_2$ such that for all $Y \in C_0$ we have*

$$i(Y) \leq m_1 \deg(Y) + m_2 p(Y)$$

**Remark.** A careful examination of the proof leads to an explicit value for $m_2$. But determining such a value for $m_1$ seems much harder.

We close this introduction by giving a consequence of Theorem 1′. To state it note that the set $A_{g,n}(\mathbb{C})$ of $\mathbb{C}$-points of $A_{g,n}$ has a natural equivalence relation on it given by isogeny: two points in $A_{g,n}(\mathbb{C})$ will be called isogenous if the corresponding abelian $\mathbb{C}$-varieties are isogenous. Each isogeny class in $A_{g,n}(\mathbb{C})$ is dense in the complex topology because it contains the image of a $\mathrm{Sp}(2g, \mathbb{Q})$-orbit on the Siegel upper half space. For any $y \in A_{g,n}(\mathbb{C})$ we denote by $I_y \subset A_{g,n}(\mathbb{C})$ the isogeny class of $y$. Then Theorem 1′ will imply the following:

**Theorem 2** *There exist a dense Kolchin open subset $C_0$ of $C(A_{g,n})$ and two positive integers $m_1$, $m_2$ such that for all $Y \in C_0$ and for any point $y \in Y(\mathbb{C})$ outside a certain countable subset of $Y(\mathbb{C})$, the set $Y(\mathbb{C}) \cap I_y$ is finite of cardinality at most $m_1 \deg(Y) + m_2 p(Y)$.*

**Remark.** As the proof will show, the countable subset of $Y(\mathbb{C})$ appearing in the above statement can be taken simply to be the set of all points in $Y(\mathbb{C})$