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CONFIGURATIONS OF REAL AND COMPLEX POLYNOMIALS

by
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This article is dedicated to the memory of Mario Raimondo.

§0. Introduction.

The purpose of this article is to give a geometric explanation of the surprising equality (cf. [C-P],[Ar1],[Ar3]) between, on one hand, the number of configurations of (complex) lemniscate generic polynomials, and , on the other hand, the number of configurations of real monic Morse polynomials with the maximal number of (real) critical points.

This discovery occurred when Arnold gave a series of talks at the Scuola Normale in 1989 on the subject of catastrophe theory, and there was somehow a bet whether there could be a geometrical correspondence between the two sets.

Afterwards, Arnold developed a quite general theory concerning the ubiquity of Euler, Bernoulli and Springer numbers (cf.[Ar1], [Ar2], [Ar3]) in the realm of singularity theory.

In this article, among other things, we prove the equality of the above two numbers by geometric methods.

It would of course be very interesting to extend the type of correspondence introduced here to a more general context, like the case of spaces of universal deformations of 0-modular isolated singularities. In a different direction, we plan to extend these type of results to the case of real algebraic functions, using the results of [B-C].

Let us explain now in some detail what are our present results.

We adopt here the notation and terminology of [C-P] and [C-W] : given a polynomial $P(z)$ we consider $|P(z)|$ as a (weak) Morse function , and we define the *big lemniscate configuration* of P to be equal to the union of the singular level sets of $|P|$ (the so called lemniscates). P is said to be *lemniscate generic* if P has distinct roots and every level set $\Gamma_c = \{z : |P(z)| = c\}$ has at most one ordinary quadratic singularity. Two big lemniscate configurations Γ_1, Γ_2 are said to be isotopic if there is a path σ in the space of diffeomorphisms of $\mathbb{C}$ such that $\sigma(0)$ is the identity and $\sigma(1)(\Gamma_1) = \Gamma_2$.

One of the main results of [C-P] was that there is a bijective correspondence between isotopy classes of big lemniscate configurations and connected components of the space $\mathcal{L}_n$ of lemniscate generic polynomials. Assume now that $P \in \mathbb{R}[z]$: then, if P is lemniscate generic, automatically all the critical points of P are real; thus, letting $(n+1)$ be the degree of P, P has n distinct real critical values which are different from zero.

Let $\mathcal{L}_n$ be the open set of complex lemniscate generic polynomials of degree $(n+1)$, let $\mathcal{L}_{n,\mathbb{R}}$ be the set of real lemniscate generic polynomials (an open set in the space of real polynomials), let finally $\mathbb{G}\mathcal{M}_n$ (which is called the "Set of generic maximally real polynomials") be the open set of real polynomials with n real and distinct critical values : thus $\mathcal{L}_{n,\mathbb{R}} \subset \mathbb{G}\mathcal{M}_n$, and every component of $\mathbb{G}\mathcal{M}_n$ is the closure (in $\mathbb{G}\mathcal{M}_n$) of a finite number of components of $\mathcal{L}_{n,\mathbb{R}}$.

If P is in $\mathbb{G}\mathcal{M}_n$ and $y_1 < ... < y_n$ are the critical points, we associate to P the sequence $u_1 = P(y_1), ..., u_n = P(y_n)$, a *snake sequence* (cf. [Da], [Ar3]), what simply means that $(-1)^i(u_i - u_{i+1})$ has constant sign.

If P is lemniscate generic and real, there is another way of ordering the critical values, namely by increasing absolute values : we let $Y_{n,\mathbb{R}} = \{(w_1, ..., w_n) \in \mathbb{R}^n : 0 < |w_1| < ... < |w_n|\}$ be the space of admissible critical values. Clearly $Y_{n,\mathbb{R}}$ has exactly 2^n connected components homeomorphic to $\mathbb{R}^n$.

Main Theorem.

(a) Each connected component of $\mathcal{L}_n$ contains exactly 2^{n+1} connected components of $\mathcal{L}_{n,\mathbb{R}}$.

(b) The number of connected components of $\mathcal{L}_{n,\mathbb{R}}$ mapping to a fixed component of $Y_{n,\mathbb{R}}$ equals the number of components of $\mathbb{G}\mathcal{M}_n$, whence the number of connected components of $\mathbb{G}\mathcal{M}_n$ equals twice the number K_n of connected components of $\mathcal{L}_n$; the number instead of components of $\mathbb{G}\mathcal{M}_n \cap$ {monic polynomials} equals K_n.

(c) (cf. Arnold [Ar1]) The number of components of $\mathbb{G}\mathcal{M}_n$ equals the number of *snake sequences* (this means, for fixed $w_1, ..., w_n$, the number of snake sequences $u_1, ..., u_n$ that can be obtained by permuting $w_1, ..., w_n$).

(d) (cf. [Ar1],[C-P]) The number of components b_n of $\mathcal{L}_{n,\mathbb{R}}$ gives rise to the following exponential generating function :

$$2\Sigma_n(b_n/n!)t^n = \int 4/(1-\sin(2t)) = 2(\sec(2t)+\tan(2t)).$$

e) the number of snake sequences equals the number of isotopy classes of lemniscate configurations multiplied by 2.

The above result is related to a curious rediscovery of Riemann's existence theorem, done by Thom in 1960 ([Thom]) In fact, in 1957 C. Davis ([Da]) showed in particular that for each choice of n distinct real numbers there is a real polynomial of degree $(n+1)$ having those as critical values (in fact, up to affine transformations in the source, a unique one for each snake sequence formed with those numbers), and a similar question was asked for complex polynomials.

Thom remarked that by Riemann's existence theorem the answer is that for each choice of n distinct complex numbers and an equivalence class of admissible monodromy there exists exactly one polynomial,up to affine transformations in the source, having those points as critical values and the given monodromy.

In this paper we link the two answers by describing explicitly, even when the branch points are not all real, the monodromies which come from real polynomials.

In fact, in [C-P] it was shown also that every big lemniscate configuration occurs for some real polynomial for which the monodromy tree (cf.[C-W]) is linear (that is, homeomorphic to a segment).

Here, in a similar vein, we establish another result (which is essential in order to establish our main theorem), which allows us to understand the lemniscate configurations which come from real polynomials as the ones obtained from "snake" linear trees (theorem B stated below is an abridged version of theorems 2.1, 2.3 and 2.12) :

Theorem B.

Given $w_1, ..., w_n \in \mathbb{R}$ with $0 < |w_1| < ... < |w_n|$, there is a canonical choice of a geometric basis of $\pi_1(\mathbb{C} - \{w_1, ..., w_n\}, 0)$ such that the real lemniscate generic polynomials P having $w_1, ..., w_n$ as critical values, correspond exactly to the monodromy trees which are "snake" linear trees.

Also, for each fixed choice of $w_1, ..., w_n$ as above, if $n \geq 4$ there is some lemniscate configuration which cannot be obtained with a real polynomial.

To get the flavour of the second statement one should remark that the monodromies which come from real monic polynomials, (whose number is $K_n \sim O((2/\pi)^n(n)!))$ are quite few compared with all the possible monodromies, whose number is $(n+1)^{n-2}$. Nevertheless, since the number of lemniscate configurations is exactly K_n, we initially hoped that there would be a bijection between the set of real monodromies and the set of lemniscate configurations.

From theorem 2.1 it is then easily seen that, fixing the (real) critical values, and a linear tree in the canonical basis, the snake condition is equivalent to the condition that the associated polynomial is real.

In this way part b) of the main theorem is proven.

Finally, the proof of a) of the main theorem is a straightforward consequence of Lefschetz' fixed points theorem, while c) follows from the quoted result of Davis, which we reprove (in 2.3) with a small precision, for the sake of completeness.

Parts d),e) follow then from a),b),c) and the results of [Ar1],[C-P].

Section 2 contains also other miscellaneous results.

In the third section we employ the branch points map used by several authors ([Da],[Lo],[Ly],[C-W],[C-P],[Ar3]) in order to give a quick proof of a generalization of Davis' theorem along the same lines. Later on, we prove in theorem 3.7 a much more precise result, namely that the monodromies of real generic polynomials are given, in a canonical basis, by trees obtained from a snake linear trees by adding, in a symmetric way, pairs of isomorphic trees