LUCA CHIANTINI
CIRO CILIBERTO

A few remarks on the lifting problem

Astérisque, tome 218 (1993), p. 95-109
<http://www.numdam.org/item?id=AST_1993__218__95_0>

A FEW REMARKS ON THE LIFTING PROBLEM

by Luca Chiantini and Ciro Ciliberto*

0. Introduction

Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, the projective space of dimension r over an algebraically closed field k of characteristic zero. If W is an irreducible variety of dimension n+m and degree s containing X, then for a general point t in the grassmannian Grass(h,r) of the h-planes in $\mathbf{P}^r$, with $h+n\geq r$, the corresponding h-plane L_t intersects X along a subvariety $X_t=X\cap L_t$ lying on the irreducible variety $W_t=W\cap L_t$ of dimension h+n+m-r and degree s.

Conversely, assume we have the following situation:

(0.1) Let X be a reduced, non-degenerate variety of dimension n in $\mathbf{P}^r$, let B be a smooth scheme and f: $B\to$Grass(h,r) a dominant smooth morphism, $h+n\geq r$. For any $t\in B$ we let L_t be the h-plane corresponding to the point $f(t)\in$ Grass(h,r). Let W in $B\times\mathbf{P}^r$ be a family of projective varieties flat over B. For $t\in B$ we let W_t be the fibre of W over t. We suppose that the general fibre W_t of W is irreducible of dimension h+n+m-r and degree s, and that for $t\in B$ one has $L_t\supseteq W_t\supseteq X_t=X\cap L_t$.

In such a situation it is not true in general that there is a variety W of dimension n+m and degree s containing X and such that $W_t=W\cap L_t$ for $t\in B$: e.g. a general plane section of an irreducible curve of degree five in $\mathbf{P}^3$ lies on a conic, whereas there are such quintic curves lying in no quadrics.

The <u>lifting problem</u> consists in looking for suitable conditions on the variety X and the family W ensuring the existence of the variety W such that $W_t=W\cap L_t$ for $t\in B$.

* Both authors have been supported by MURST and CNR of Italy. The research has been performed in the framework of Europroj's project "Hyperplane sections".

The problem has been first considered for the case of curves in $\mathbf{P}^3$, i.e. n=1, r=3, by Laudal [5], who gave a solution later refined by Gruson and Peskine [3]. Gruson-Peskine's result asserts that if X is a reduced, irreducible curve of degree d in $\mathbf{P}^3$, whose general plane section lies on some curve Γ of degree s, and if $d>s^2+1$, then X lies on a surface of degree s whose general plane section is Γ. Curves arising from sections of a null-correlation bundle show that the bound $d>s^2+1$ is sharp (see [3], [12]). More results on the lifting problem, especially for curves in $\mathbf{P}^3$, have been found by Strano with a purely algebraic approach relating the lifting problem to the syzigies of the resolution of the ideal of X_t (see [11], [9] and [6]).

Inspired by Gruson-Peskine's result mentioned above, we will restrict ourselves to the search of a function D(s,h,r,n,m) such that, if (0.1) holds, the lifting problem has a positive answer for d>D(s,h,r,n,m). And one could be so optimistic to try to find an <u>optimal</u> such function, i.e. a function D(s,h,r,n,m) with the above properties and such that there are counterexamples to the lifting problem for d≤D(s,h,r,n,m), e.g. in Gruson-Peskine's case (h=2, r=3, n=m=1) the <u>optimal</u> function is $D(s)=s^2+1$. The question, if one puts in this form, makes sense only if dim W_t=dim X_t+1, i.e. only if m=1 (see however § 3). Consider in fact the following:

<u>Example</u>: Let V be a smooth projection of the Veronese surface in $\mathbf{P}^4$, which is known to be not contained in any quadric 3-fold. Let X be an irreducible curve cut out on V by a hypersurface of degree d>3. By the theorem of Bezout X does not lie on any quadric 3-fold in $\mathbf{P}^4$, whereas its general hyperplane section is contained on a quartic rational curve, hence it does lie on a quadric surface in $\mathbf{P}^3$.

Hence in the present paper we will mainly restrict our attention to the case m=1, and we will determine a function D(s,h,r,n) such that if X has dimension n and degree d>D(s,h,r,n), and if there is a family W as in (0.1) with m=1, then there is a variety W of dimension n+1 such that $W_t=W\cap L_t$ for $t\in U$. The proof makes use of the differential-geometric concepts of <u>foci</u> and of <u>focal locus</u> for families of projective varieties, a classical notion firstly systematised by C. Segre [10] for families of linear subspaces and recently extended in [1] to any family of projective varieties. Similar ideas are already present in implicit form in [3]. We collect in § 1 all basic facts about foci and focal loci of a family which we need in the sequel. In § 2 we show that, if the lifting problem for X and the family W as in (0.1) with m=1 has a negative answer, then the points of X either lie in the focal locus of W or X_t lies in the singular locus of W_t for t a general point in B. Then by estimating the degrees of these loci, we prove the following:

Theorem (0.2).- Let X be a reduced, non-degenerate, projective subvariety of dimension n and degree d in $\mathbf{P}^r$ and let us suppose there is a family W as in (0.1) with m=1. If

$$d>D(s,h,r,n):=(r+h-3)s+k(k-1)(r-n-1)+2ek-2$$

where $s-1=k(r-n-1)+e$, $0\leq e<r-n-1$, then the image W of W in $\mathbf{P}^r$ is a variety of dimension n+1 and degree s, containing X and such that $W_t=W\cap L_t$ for $t\in B$.

Our function D(s,h,r,n) is not optimal in general. Slight improvements can be obtained in some cases with a more detailed analysis in the same vein of our proof below: for example the case of codimension two, n=r-2, has been recently carefully investigated by E. Mezzetti [7], whose result fully generalizes Gruson-Peskine's theorem to the case $r\leq 5$. She also makes a nice conjecture on the optimal function D(s,r-1,r,r-2). However we point out that, although in general not optimal, our function D(s,h,r,n) is asymptotically optimal. Indeed for instance in the case of curves we have that $D(s,r):=D(s,r-1,r,1)=[s^2/(r-2)]+o(s)$ and we find in § 3 curves X in $\mathbf{P}^r$ of degree $d=d(s)>>0$ with $d<D(s,r)$ but with $D(s,r)=d(s)+o(s)$, for which the lifting fails. These curves, as well as the curves in $\mathbf{P}^3$ achieving Gruson-Peskine's bound, are obtained as sections of suitable rank two vector bundles on certain rational normal scrolls. At the end of § 3 we will also briefly discuss an extension of theorem (0.2) to the case m>1.

In conclusion we want to mention that our approach via the focal loci has unexpected close relationships with Strano's algebraic approach mentioned above. We do not exploit this in the present paper, but we hope to come back on this subject in the future.

1. Generalities on foci.

In this section we let:

B be a non singular scheme of dimension b

W inside $B\times\mathbf{P}^r$ be a family, flat over B, of irreducible projective varieties of dimension w

V be a desingularization of W

After having shrinked B we may assume that V is flat over B, with smooth and irreducible fibres. Indeed, we may assume that for $t\in B$, the fibre V_t of $\mathsf{V}\to B$ over t is a desingularization of the corresponding fibre W_t of $\mathsf{W}\to B$.

The natural morphism u: $\mathsf{V}\to B\times\mathbf{P}^r$ yields the map of sheaves du: $T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}$ which is generically injective, and therefore injective, since T_{V} is locally free. The cokernel of du is, by definition, the normal sheaf N_u to the map u, thus we have the exact sequence

(1.1) $$0\to T_{\mathsf{V}}\to u^*T_{B\times\mathbf{P}^r}\to N_u\to 0$$

and we notice that, in general, N_u is not necessarily torsion free.

We let p: $B\times\mathbf{P}^r\to B$ and q: $B\times\mathbf{P}^r\to\mathbf{P}^r$ be the projections. Then we have another natural map dq: $u^*T_{B\times\mathbf{P}^r}\to u^*q^*T_{\mathbf{P}^r}$ which is surjective. The kernel of dq is a locally free sheaf T(q) of rank b on V and we have the exact sequence

(1.2) $$0\to T(q)\to u^*T_{B\times\mathbf{P}^r}\to u^*q^*T_{\mathbf{P}^r}\to 0$$

The above sequences (1.1) and (1.2) fit into the commutative exact diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & L \arrow[r] \arrow[d] & T(q) \arrow[r, "\lambda"] \arrow[d] & N_u \arrow[d, equal] & \\
0 \arrow[r] & T_V \arrow[r] \arrow[d, "\partial"'] & u^*T_{B\times\mathbf{P}^r} \arrow[r] \arrow[d] & N_u \arrow[r] & 0 \\
 & u^*q^*T_{\mathbf{P}^r} \arrow[r, equal] & u^*q^*T_{\mathbf{P}^r} \arrow[d] & & \\
 & & 0 & &
\end{tikzcd}$$

where ∂ is the differential of the map $q\circ u$, λ is the characteristic map for the family V and L is the kernel of λ.

Since we are in characteristic 0, q is smooth at the general point of W. So if we set w_o=dim q(W), then we have

$$\text{rk }\partial=w_o,\ \text{rk }L=\text{rk }T_V-w_o=b+w-w_o,\ \text{rk }\lambda=w_o-w$$

where of course $w_o-w=\dim q(W)-w\geq 0$.

Next we consider the restriction of λ to a general fibre of $V\to B$. Take $t\in B$ and let V_t be the corresponding fibre of $V\to B$. Let U be an affine open neighborhood of t in B over which T_B trivializes. Then over $p^{-1}(U)$ the map dq: $T_{B\times\mathbf{P}^r}\to q^*T_{\mathbf{P}^r}$ has a trivial kernel. Accordingly T(q) also trivializes over $V=u^{-1}p^{-1}(U)$, hence we have an isomorphism

(1.3) $$T(q)_{|V}\cong \mathcal{O}_V^{\,b}$$

Now we denote by N_t the normal sheaf to the induced map $u_t=q\circ u_{|V_t}: V_t\to\mathbf{P}^r$, and we prove the following basic:

Proposition (1.4).- One has $N_{u\,|\,V_t}\cong N_t$.

Proof. Consider the following commutative exact diagram: