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1 Introduction

The notion of *quantum cohomology ring* of a Kähler manifold V naturally appears in theoretical physics in the consideration of the so called *topological sigma model* associated with V ([16], 3a-b). If the canonical line bundle $\mathcal{K}_V$ of V is negative, then one recovers the multiplicative structure of the quantum cohomology ring of V from the intersection theory on the moduli space $\mathcal{I}_\lambda$ of holomorphic mappings f of the Riemann sphere $f : S^2 \cong \mathbf{CP}^1 \to V$ where λ is the homology class in $H_2(V, \mathbf{Z})$ of $f(\mathbf{CP}^1)$.

If the canonical bundle $\mathcal{K}_V$ is trivial, the quantum cohomology ring was considered by C. Vafa as an important tool for explaining the mirror symmetry for Calabi-Yau manifolds [15].

The quantum cohomology ring $QH_\varphi(V, \mathbf{C})$ of a Kähler manifold V, unlike the ordinary cohomology ring, have the multiplicative structure which depends on the class φ of the Kähler $(1,1)$-form corresponding to a Kähler metric g on V. When we rescale the metric $g \to tg$ and put $t \to \infty$, the quantum ring becomes the classical cohomology ring. For example, for the topological sigma model on the complex projective line $\mathbf{CP}^1$ itself, the classical cohomology ring is generated by the class x of a Kahler $(1,1)$-form, where x satisfies the quadratic equation

$$x^2 = 0, \tag{1}$$

while the quantum cohomology ring is also generated by x, but the equation satisfied by x is different:

$$x^2 = \exp(-\int_\lambda \varphi), \tag{2}$$

λ is a non-zero effective 2-cycle. Similarly, the quantum cohomology ring of d-dimensional complex projective space is generated by the element x satisfying the equation

$$x^{d+1} = \exp(-\int_\lambda \varphi). \tag{3}$$

The main purpose of this paper is to construct and investigate the quantum cohomology ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ of an arbitrary smooth compact d-dimensional toric manifold $\mathbf{P}_\Sigma$ where φ is an element of the ordinary second cohomology group $H^2(\mathbf{P}_\Sigma, \mathbf{C})$. Since all projective spaces are toric manifolds, we obtain a generalization of above examples of quantum cohomology rings.

According to the physical interpretation, a quantum cohomology ring is a closed operator algebra acting on the fermionic Hilbert space. For example, the equation (3) one should better write as an equations for a linear operator $\mathcal{X}$ corresponding to the cohomology class x:

$$\mathcal{X}^{d+1} = \exp(-\int_\lambda \varphi)id. \tag{4}$$

It is convenient to define quantum rings by polynomial equations among generators.

Definition 1.1 Let

$$h(t,x) = \sum_{n\in\mathcal{N}} c_n(t)x^n$$

be a one-parameter family of polynomials in the polynomial ring $\mathbf{C}[x]$, where $x = \{x_i\}_{i\in I}$ is a set of variables indexed by I, t is a positive real number, $\mathcal{N}$ is a fixed finite set of exponents. We say that the polynomial

$$h^\infty(x) = \sum_{n\in\mathcal{N}} c_n^\infty x^n$$

is the limit of the family $h(t,x)$ as $t \to \infty$, if the point $\{c_n^\infty\}_{n\in\mathcal{N}}$ of the $(\mid \mathcal{N} \mid -1)$-dimensional complex projective space is the limit of the one-parameter family of points with homogeneous coordinates $\{c_n(t)\}_{n\in\mathcal{N}}$.

Definition 1.2 Let R_t be a one-parameter family of commutative algebras over $\mathbf{C}$ with a fixed set of generators $\{r_i\}$, $t \in \mathbf{R}_{>0}$. We denote by J_t the ideal in $\mathbf{C}[x]$ consisting of all polynomial relations among $\{r_i\}$, i.e., the kernel of the surjective homomorphism $\mathbf{C}[x] \to R_t$. We say that the ideal J^∞ is the limit of J_t as $t \to \infty$, if any one-parameter family of polynomials $h(t,x) \in J_t$ (as in 1.1) has a limit, and J^∞ is generated as $\mathbf{C}$-vector space by all these limits. The $\mathbf{C}$-algebra

$$R^\infty = \mathbf{C}[x]/J^\infty$$

will be called the *limit* of R_t.

Remark 1.3 In general, it is not true that if $J^\infty = \lim_{t\to\infty} J_t$, and J_t is generated by a finite set of polynomials $\{h_1(t,x)\ldots,h_k(t,x)\}$, then J^∞ is generated by the limits $\{h_1^\infty(x),\ldots,h_k^\infty(x)\}$. The limit ideal J^∞ is generated by the limits $h_i^\infty(x)$ *only if* the set of polynomials $\{h_i(t,x)\}$ form a *Gröbner-type basis* for J_t.

In this paper, we establish the following basic properties of quantum cohomology rings of toric manifolds:

I : If φ is an element in the interior of the Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$, then there exists a limit of $QH^*_{t\varphi}(\mathbf{P}_\Sigma, \mathbf{C})$ as $t \to \infty$, and this limit is isomorphic to the ordinary cohomology ring $H^*(\mathbf{P}_\Sigma, \mathbf{C})$ (Corollary 5.5).

II : Assume that two smooth projective toric manifolds $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are isomorphic in codimension 1, for instance, that $\mathbf{P}_{\Sigma_1}$ is obtained from $\mathbf{P}_{\Sigma_2}$ by a flop-type birational transformation. Then the natural isomorphism $H^2(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong H^2(\mathbf{P}_{\Sigma_2}, \mathbf{C})$ induces the isomorphism between the quantum cohomology rings

$$QH^*_\varphi(\mathbf{P}_{\Sigma_1}, \mathbf{C}) \cong QH^*_\varphi(\mathbf{P}_{\Sigma_2}, \mathbf{C})$$

(Theorem 6.1). We notice that ordinary cohomology rings of $\mathbf{P}_{\Sigma_1}$ and $\mathbf{P}_{\Sigma_2}$ are not isomorphic in general.

III : Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ belongs to the closed Kähler cone $K(\mathbf{P}_\Sigma) \subset H^2(\mathbf{P}_\Sigma, \mathbf{C})$. Then the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ is isomorphic to the Jacobian ring of a Laurent polynomial $f_\varphi(X)$ such that the equation $f_\varphi(X) = 0$ defines an affine Calabi-Yau hypersurface Z_f in the d-dimensional algebraic torus $(\mathbf{C}^*)^d$ where Z_f is *mirror symmetric* with respect to Calabi-Yau hypersurfaces in $\mathbf{P}_\Sigma$ (Theorem 8.4). Here by the *mirror symmetry* we mean the correspondence, based on the polar duality [6], between families of Calabi-Yau hypersurfaces in toric varieties.

The properties II and III give a general view on the recent result of P. Aspinwall, B. Greene, and D. Morrison [3] who have shown, for a family of Calabi-Yau 3-folds W that their quantum cohomology ring $QH^*_\varphi(W, \mathbf{C})$ does not change under a flop-type birational transformation (see also [1, 2]).

IV: Assume that the first Chern class $c_1(\mathbf{P}_\Sigma)$ of $\mathbf{P}_\Sigma$ is divisible by r, i.e., there exists an element $h \in H^2(\mathbf{P}_\Sigma, \mathbf{Z})$ such that $c_1(\mathbf{P}_\Sigma) = rh$. Then $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has a natural $\mathbf{Z}/r\mathbf{Z}$-grading (Theorem 5.7). We remark that the ring $QH^*_\varphi(\mathbf{P}_\Sigma, \mathbf{C})$ has no $\mathbf{Z}$-grading.

The paper is organized as follows. In Sections 2-4, we recall a definition of toric manifolds and standard facts about them. In Section 5, we define the quantum cohomology ring of toric manifolds and prove their properties.

In Section 6, we consider examples of the behavior of quantum cohomology rings under elementary birational transformations such as blow-ups and flops, we also consider the case of singular toric varieties. In Section 7, we give an combinatorial interpretation of the relation between the quantum cohomology rings and the ordinary cohomology rings. In Section 8, we show that the quantum cohomology ring can be interpreted as a Jacobian ring of some Laurent polynomial. Finally, in Section 9, we prove that our quantum cohomology rings coincide with the quantum cohomology rings defined by sigma models on toric manifolds.

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2 A definition of compact toric manifolds

Toric varieties were considered in full generality in [9, 11]. For the general definition of toric variety which includes affine and quasi-projective toric varieties with singularities, it is more convenient to use the language of schemes. However, for our purposes, it will be sufficient to have a simplified more classical version of the definition for *smooth and compact toric varieties* over $\mathbf{C}$. This approach to compact toric manifolds was first proposed by M. Audin [4], and developed by D. Cox [8].

In order to obtain a d-dimensional compact toric manifold V, we need a combinatorial object Σ, a *complete fan of regular cones*, in a d-dimensional vector space over $\mathbf{R}$.

Let N, $M = \mathrm{Hom}(N, \mathbf{Z})$ be dual lattices of rank d, and $N_{\mathbf{R}}$, $M_{\mathbf{R}}$ their $\mathbf{R}$-scalar extensions to d-dimensional real vector spaces.

Definition 2.1 A convex subset $\sigma \subset N_{\mathbf{R}}$ is called a *regular k-dimensional cone* ($k \geq 1$) if there exist k linearly independent elements $v_1, \ldots, v_k \in N$ such that

$$\sigma = \{\mu_1 v_1 + \cdots + \mu_k v_k \mid \mu_i \in \mathbf{R}, \mu_i \geq 0\},$$

and $\{v_1, \ldots, v_k\}$ is a subset of some $\mathbf{Z}$-basis of N. In this case, we call $v_1, \ldots, v_k \in N$ the *integral generators of σ*.