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Astérisque, tome 222 (1994), p. 135-199
<http://www.numdam.org/item?id=AST_1994__222__135_0>

COMPENSATION OF SMALL DENOMINATORS AND RAMIFIED LINEARISATION OF LOCAL OBJECTS

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1. REMINDER ABOUT LOCAL OBJECTS. THE THREE MAIN ANALYTIC FACTS.

By *local objects* we will understand either *local analytic vector fields* (or *fields* for short) on $\mathbf{C}^\nu$ at 0 :

$$X = \sum_{i=1}^{\nu} X_i(x)\partial_{x_i} \qquad (X_i(x) \in \mathbf{C}\{x\};\ X_i(0) = 0) \tag{1.1}$$

or *local analytic selfmappings* (or *diffeos*, short for diffeomorphisms) of $\mathbf{C}^\nu$ with 0 as fixed point :

$$f : x_i \mapsto f_i(x) \qquad (i = 1, \dots, \nu) \qquad (f_i(x) \in \mathbf{C}\{x\};\ f(0) = 0) \tag{1.2}$$

or again, equivalently, the related *substitution operators* :

$$F : \varphi \mapsto F.\varphi \overset{\text{def}}{=} \varphi \circ f \qquad (\varphi(x)\,\text{and}\,\varphi \circ f(x) \in \mathbf{C}\{x\}) \tag{1.3}$$

Throughout, we will assume *diagonalisability* of the linear part and work with (analytic) *prepared forms* of the object on hand. That is to say, we will deal with vector fields given by :

$$X = X^{\text{lin}} + \sum \mathbb{B}_n \tag{1.4}$$

$$X^{\text{lin}} = \sum \lambda_i x_i \partial_{x_i} \tag{1.4'}$$

$$\mathbb{B}_n = \ \text{homogeneous part of degree } n = (n_1, \dots, n_\nu) \qquad \text{with } n_i \geq -1 \tag{1.4''}$$

and with diffeos given by :

$$F = \{1 + \sum \mathbb{B}_n\} F^{\text{lin}} \tag{1.5}$$

$$F^{\text{lin}} \varphi(x_1, \dots, x_\nu) \overset{\text{def}}{=} \varphi(\ell_1 x_1, \dots, \ell_\nu x_\nu) \tag{1.5'}$$

$$\mathbb{B}_n = \ \text{homogeneous part of degree } n = (n_1, \dots, n_\nu) \quad \text{with } n_i \geq -1 \tag{1.5''}$$

Of course, n-homogeneity means that for each monomial $x^m = x_1^{m_1} \dots x_\nu^{m_\nu}$:

(1.6) $$\mathbf{B}_n.x^m = \beta_{n,m}x^{n+m} \qquad \text{with } \beta_{n,m} \in \mathbf{C}.$$

Note that, for any given $\mathbf{B}_n$, *at most* one component n_i may assume the value -1.

The scalars λ_i and ℓ_i are the object's *multipliers*. Together, they constitute its spectrum. If the spectrum is "random", the object turns out to be formally and even *analytically linearisable* (see below) and that about ends the matter, as far as the local study is concerned. For interesting problems to arise, at least one of three specific complications C_1, C_2, C_3 (see below) must come into play. Then numerous difficulties, mostly due to divergence, have to be sorted out. Yet the remarkable thing is that three easy, formal statements F_1, F_2, F_3 (F for formal) and three non-trivial, analytic theorems A_1, A_2, A_3 (A for analytic) suffice, between themselves, to give a fairly comprehensive picture of the whole situation. In this paper, we shall be mainly concerned with statement A_3 about the *effective, ramified linearisation of local objects*. Nonetheless, both for completeness and orientation, we shall begin with a brief review of all six statements. But first, we list the three "complications".

C_1. **Resonance.**

For a vector field, this means additive resonance of the λ_i :

(1.7) $$\sum_{i=1}^{\nu} m_i\, \lambda_i = 0$$ or (1.7′) $$\sum_{i=1}^{\nu} m_i\lambda_i = \lambda_j \quad (m_i \in \mathbf{N})$$

and for a diffeo it means multiplicative resonance of the ℓ_i :

(1.8) $$\prod_{i=1}^{\nu} (\ell_i)^{m_i} = 1$$ or (1.8′) $$\prod_{i=1}^{\nu} (\ell_i)^{m_i} = \ell_j \ (m_i \in \mathbf{N})$$

C_2. **Quasiresonance.**

This means that among all the non-vanishing expressions $\alpha(m) = < m, \lambda >$ or $\alpha(m) = \ell^m - 1$ (with all $m_i \geq 0$ except at most one that may be $\equiv -1$) there is a subinfinity that tends to 0 "abnormally" fast, thus violating the two equivalent diophantine conditions :

(1.9) $$S \stackrel{\text{def}}{=} \sum 2^{-k} \log(1/\varpi(2^k)) \ < +\infty \qquad \text{(A.D. Bruno)}$$

(1.9′) $$S^* \stackrel{\text{def}}{=} \sum k^{-2} \log(1/\varpi(k)) \ < +\infty \qquad \text{(H. Rüssmann)}$$

with $\varpi(k) = \inf |\alpha(m)|$ for $m_1 + \ldots m_\nu \leq k$. (Clearly, $1/2 \leq S^*/S \leq 2$).

C_3. **Nihilence.**

It amounts to the existence of a "first integral" in the form of a power series H :

$$X.H(x) \equiv 0 \text{ or } H(f(x)) \equiv H(x) \text{ with } H(x) \in \mathbb{C}[[x]] = \mathbb{C}[[x_1, \ldots, x_\nu]] \tag{1.10}$$

along with the existence of small denominators :

$$\inf |\alpha(m)| = 0 \text{ for } \alpha(m) \neq 0. \tag{1.11}$$

Note that conditions (1.10) presupposes *resonance*, but that condition (1.11) differs from (1.9) in that it involves no arithmetical condition.

Resonance is fairly common, if only because it includes all diffeos tangent to the identity map, for which indeed $\ell_1 = \ell_2 = \ldots \ell_\nu = 1$. *Quasiresonance* is decidedly exceptional in single objects, but becomes inescapable when one studies parameter-dependent families of objects. *Nihilence* is common with volume-preserving or symplectic objects, where it may occur, respectively, from dimension 3 and 4 onwards.

All three complications may coexist. They may even occur in layers. Indeed, whenever the ordinary or first-level multipliers of an object X or F are involved in *multiple resonance*, there is a natural notion of reduced object X^{red} or F^{red} acting on the algebra of resonant monomials, and endowed with its own multipliers (second-level multipliers), which may in turn give rise to second-level resonance, quasiresonance or nihilence; and so forth. This daunting multiplicity of cases and subcases makes the existence of universally valid statements like A_1, A_2, A_3 (infra) all the more remarkable. But first let us go through the formal statements F_1, F_2, F_3 which, though fairly trivial, will clear the ground for A_1, A_2, A_3 and settle some useful terminology.

F$_1$. *In the absence of resonance, a local object is formally linearisable.*

The proof is straightforward. Indeed, inductive coefficient identification yields formal, entire changes of coordinates :

$$h^{\text{ent}} : x \mapsto y \quad \text{with} \quad y_i = h_i^{\text{ent}}(x) = x_i\{1 + \ldots\} \tag{1.12}$$

$$k^{\text{ent}} : y \mapsto x \quad \text{with} \quad x_i = k_i^{\text{ent}}(y) = y_i\{1 + \ldots\} \tag{1.13}$$

which take us from the given analytic chart $x = (x_i)$ to a formal chart $y = (y_i)$ where the object reduces to its linear part X^{lin} or F^{lin}. But to pave the way for the forthcoming analytic study, we require explicit expansions for h^{ent} and k^{ent}, or rather for the corresponding formal substitution operators Θ_{ent} and Θ_{ent}^{-1}. We use the variables x_i throughout :

$$\Theta_{\text{ent}}\ \varphi(x) \overset{\text{def}}{\equiv} \varphi \circ h^{\text{ent}}(x) \qquad (\varphi(x), \varphi \circ h^{\text{ent}}(x) \in \mathbb{C}[[x]]) \tag{1.14}$$