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Introduction

This paper deals with two closely related topics:

1. Finitely smooth normal forms for local families.
2. Bifurcations of polycycles of few- and many- parameter families. Here "few" is "no greater than 3"

The exposition is the summary of two large paper [I,Y3] and [K,S] which are to be published in the forthcoming book [I]. Therefore all the proofs are brief in this text; there detailed exposition would be found in the book, quoted above.

It appears, that for the study of nonlocal behavior of the orbits of vector field from the *topological* point of view, the *smooth* normal forms of vector field near singular points are necessary. For instance, consider a separatrix loop of a hyperbolic saddle (Figure 1).

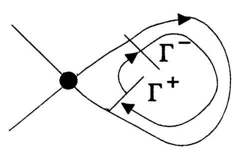

FIGURE 1

We want to know, wether the positive semiorbits winging inside the separatrix loop come *to* or *off* this loop. The topological normal form of the field near the saddle is one and the same for all the fields and give no information on the subject; it is

$$\dot{x} = x, \quad \dot{y} = -y$$

Meanwhile, the smooth normal form in the nonresonant case is

$$\dot{x} = \lambda_1, \quad \dot{y} = -\lambda_2 y$$

$\lambda_1 > 0, -\lambda_2 < 0$ are the eigenvalues of the singular point. The *correspondence map* of the entrance semitransversal Γ^+ onto the exit one Γ^- is equal to

$$\Delta(x) = x^\lambda, \lambda = \lambda_2/\lambda_1$$

Suppose $\lambda \neq 1$. Then the correspondence map in the small neighborhood of O on Γ^+ has a large Lipshitz constant; in the case $\lambda > 1$ this constant tends to zero as the neighborhood contracts to a point. The smooth map from Γ^+ to Γ^- along the orbits cannot neutralize this contraction; therefor in the case $\lambda > 1$, the separatrix loop is orbitally stable from inside. In the same way, it is unstable if $\lambda < 1$.

The example motivates the study of smooth normal forms of local families.

On the other hand, the bifurcations of polycycles are closely related with to Hilbert 16^{th} problem, as is discussed below.

§1. NUMBERS RELATED TO THE HILBERT 16^{th} PROBLEM.

Consider a family of differential equations

$$\frac{dy}{dx} = \frac{P_n(x,y)}{Q_n(x,y)} \tag{1}$$

where P_n and Q_n are polynomials of degree no larger than the fixed constant n. The following definition is popular in the survey literature.

Definition 1. The Hilbert number $H(n)$ is the maximal possible number of limit cycles of the equation of the family (1).

It is obvious, that $H(1) = 0$. Indeed, a linear vector field has no limit cycles at all.

Nothing is known about the numbers $H(2)$; its mere existence is an open problem.

One can figure out, why Hubert has chosen the family (1) for the study of limit cycles. In the end of the last century polynomial families gave probably the only natural example of finite parameter families of vector fields.

Now, when the mode and viewpoints have reasonably changed, generic finite parameter families became respectful. Therefore a smooth version of the Hilbert 16^{th} problem may be stated; it is written between the lines of some text due to Arnold [AAIS].

Hilbert-Arnold conjecture. *The number of limit cycles of the equation of the typical finite parameter family* (here and below "family" means " $\mathbb{C}^\infty$ family of vector fields in S^2 ") *with the compact base is uniformly bounded with respect to the parameter.*

This conjecture is closely related to some nonlocal bifurcation problem. We will first state it and then recall necessary natural definitions.

Conjecture. *Cyclicity of any polycycle appearing in the typical finite parameter family is finite.*

Definition 2. A polycycle is a finite union of singular point and continual phase curves of the field which is connected and cannot be contracted along itself to any proper subset.

A limit cycle is generated by a polycycle γ in the family

$$\dot{x} = v(x, \epsilon), \quad x \in S^2, \epsilon \in B \subset \mathbb{R}^k$$

if the path $\epsilon(t)$ in the parameter space exits such that for any $t \in (0, 1]$ the equation corresponding to $\epsilon(t)$ has a limit cycle $l(t)$, continuously depending on the parameter t, $l(1) = l$, and

$$l(t) \to \gamma \text{ as } t \to 0$$

in sense of the Hausdorff distance.

Cyclicity of the polycycle in the family is the maximal number of limit cycles generated by this polycycle and corresponding to the parameter value, close to the critical one; the last corresponds to the equation with the polycycle.

Theorem (Roussarie). *The equations of the family with the compact base and the polycycles having finite cyclicity only have a uniformly bounded number of limit cycles.*

Therefore the last Conjecture implies the Hilbert-Arnold one. Some *bifurcation numbers* related to these Conjectures, are naturally defined.

Recall that a singular point of a planar vector field is called *elementary* if it has at least one nonzero eigenvalue. A polycycle is called elementary if all its vertexes are elementary.

Definition 3. $B(n)$ is the maximal number of limit cycles which can be generated by a polycycle met in a typical n-parameter family.

$E(n)$ is the maximal numbers of limit cycles which can be generated by an *elementary* polycycle in a typical n-parameter family.

$C(n)$ is the maximal number of limit cycles which can bifurcate in a typical n-parameter family from *all* the polycycles of the field, corresponding to the "critical" value of the parameter.

Conjecture. *$B(n)$ exists and is finite for any n.*

This Conjecture is stronger then Hilbert-Arnold one.

§2. STATEMENTS OF RESULTS.

Theorem 1 (Ilyashenko & Yakovenko). *For any n the number $E(N)$ exists.*

Theorem 2 (Kotova). $C(3) = \infty$.

This means that for any N one can find a generic 3-parameter family, in which some differential equation generates more than N limit cycles.

Moreover, a complete list of polycycles which can generate limit cycles and appear in generic 2 and 3- families is given; this is so called "Zoo of Kotova", Table 1 below.

Theorem 3. $B(2) = 2$.

Theorem 4. $C(2) = 3$.

Last two theorems are due to Grosowskii, Druzkova, Chelubeev and Seregin.

Theorem 5 (Stanzo). *For generic three parameter families there is a countable number of topologicaly nonequivalent germs of bifurcation diagrams.*

In this form the Theorem 5 is an easy consequence of the Theorem 2. In fact Stanzo describes the topological and even the smooth structure of bifurcation diagrams for unfoldings of the phase portrait called "lips"(Figure 2), and constructs the invariants of the topological structure of these diagrams.

As a by product of this study a *generalized Legendre duality* is found.

Comments to the Table 1. In the Table 1 all the polycycles which can appear in the generic 2 and 3 parameter families are presented. For sure, "all" means "all equivalence clases"; the equivalence relation is a following. Two polycycles are equivalent, if they have diffeomorphic neighborhoods in $\mathbb{R}^2$ and a diffeomorphism of one of them to another exists which transforms one