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V. P. KOSTOV

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REGULAR LINEAR SYSTEMS ON CP^1 AND THEIR MONODROMY GROUPS

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1. INTRODUCTION

1.1

A meromorphic linear system of differential equations on $\mathbf{C}P^1$ can be presented in the form

$$\dot{X} = A(t)X \tag{1}$$

where $A(t)$ is a meromorphic on $\mathbf{C}P^1$ $n \times n$ matrix function, " $\cdot$ " $\equiv d/dt$. Denote its poles $a_1, \dots, a_{p+1}$, $p \geq 1$. We consider the dependent variable X to be also $n \times n$-matrix.

Definition. System (1) is called *fuchsian* if all the poles of the matrix-function $A(t)$ are of first order.

Definition. System (1) is called *regular* at the pole a_j if in its neighbourhood the solutions of the system are of moderate growth rate, i.e.

$$||X(t - a_j)|| = 0\left(|t - a_j|^{N_j}\right), \quad N_j \in \mathbf{R},\ \ j = 1, \dots, p+1$$

Here $|| \cdot ||$ denotes an arbitrary norm in $gl(n, \mathbf{C})$ and we consider a restriction of the solution to a sector with vertex at a_j and of a sufficiently small radius, i.e. not containing other poles of $A(t)$. Every fuchsian system is regular, see [1]. The restriction to a sector is essential, if we approach the pole along a spiral encircling it sufficiently fast, then we can obtain an exponential growth rate for $||X||$.

Two systems (1) with the same set of poles are called *equivalent* if there exists a meromorphic transformation (equivalency) on $\mathbf{C}P^1$

$$X \mapsto W(t)X \tag{2}$$

with $W \in \mathcal{O}(\mathbf{C}P^1\backslash\{a_1,...,a_{p+1}\})$ and $\det W(t) \neq 0$ for $t \in \mathbf{C}P^1\backslash\{a_1,...,a_{p+1}\}$ which brings the first system to the second one. A transformation (2) changes system (1) according to the rule

$$A(t) \to -W^{-1}(t)\dot{W}(t) + W^{-1}(t)A(t)W(t) \tag{3}$$

1.2

The monodromy group of system (1) is defined as follows: fix a point $a \neq a_j$ for $j = 1,...,p+1$, fix a matrix $B \in GL(n,\mathbf{C})$ and fix p closed contours on $\mathbf{C}P^1$ beginning at the point a each of which contains exactly one of the poles a_j of system (1), see Fig. 1. The *monodromy operator* corresponding to such a contour is the linear operator mapping the matrix B onto the value of the analytic continuation of the solution of system (1) which equals B for $t = a$ along the contour encircling a_j; we assume that all the contours are positively orientated. Monodromy operators act on the right, i.e. we have $B \mapsto BM_j$. The monodromy operators $M_1,...,M_p$ corresponding to $a_1,...,a_p$ generate the *monodromy group* of system (1) which is a presentation of the fundamental group $\pi_1(\mathbf{C}P^1\backslash(a_1,\ldots,a_{p+1})$ into $\mathrm{GL}(n,\mathbf{C})$; we have

$$M_{p+1} = (M_1....M_p)^{-1} \tag{4}$$

for a suitable ordering of the points a_j and the contours, see Fig. 1.

It is clear that

1. the monodromy group is defined up to conjugacy due to the freedom in choosing the point a and the matrix B.

2. the monodromy groups of equivalent systems are the same.

The monodromy group of a regular system is its only invariant under meromorphic equivalence.

Capital Latin letters (in most cases) denote matrices or their blocks; by I we denote $\mathrm{diag}(1,\ldots,1)$.

1.3

It is natural to consider $GL(n,\mathbf{C})^p$ as the space of monodromy groups of regular systems on $\mathbf{C}P^1$ with $p+1$ prescribed poles (because the operators $M_1,\ldots,M_p$ define the monodromy group of system (1)). Condition (4) allows one to consider M_{p+1} as an analytic matrix-function defined on $GL(n,\mathbf{C})^p$. Of course, in a certain sense, $M_1,\ldots,M_{p+1}$ are 'equal', i.e. anyone of them can play the role of

M_{p+1}. We define *an analytic stratification* of $(GL(n,\mathbf{C}))^p$ by the Jordan normal forms of the operators $M_1,\ldots,M_{p+1}$ and the possible reducibility of the group $\{M_1,\ldots,M_p\}$. Fixing the Jordan normal form of $M_1,\ldots,M_p$ is equivalent to restricting the matrix-function $M_{p+1} = (M_1\ldots M_p)^{-1}$ to a smooth analytic subvariety of $GL(n,\mathbf{C})^p$, but if we want to fix the one of M_{p+1} as well, then we a priori can say nothing about the smoothness of the subset of $GL(n,\mathbf{C})^p$ (called *superstratum*) obtained in this way. The basic aim of this paper is to begin the study of the stratification of $GL(n,\mathbf{C})^p$ and the smoothness of the strata and superstrata.

Throughout the paper 'to fix the Jordan normal form' means 'to define the multiplicities of the eigenvalues and the sizes and numbers of Jordan blocks corresponding to each of them', but *not* to fix the eigenvalues as well; this is called 'to fix the orbit'.

2 The stratification of the space of monodromy groups

Definition. Let the group $\{M_1,\ldots,M_p\} \subset GL(n,\mathbf{C})$ be conjugate to one in block-diagonal form, the diagonal blocks (called *big blocks*) being themselves block upper-triangular; their block structure is defined by their diagonal blocks (called *small blocks*). The restriction of the group to everyone of the small blocks is assumed to be an irreducible matrix group of the corresponding size. The sizes of the big and small blocks are correctly defined modulo permutation of the big blocks (if we require that the sizes of the big blocks are the minimal possible) and define the *reducibility type* of the group.

Example : The reducibility type $\begin{pmatrix} A & B & 0 \\ 0 & C & 0 \\ 0 & 0 & Q \end{pmatrix}$ has two big ($\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$ and Q) and three small blocks (A, C and Q).

Definition. A *stratum* of $GL(n,\mathbf{C})$ is its subset of matrices with one and the same Jordan normal form. A group $\{M_1,\ldots,M_p\} \subset GL(n,\mathbf{C})$ defines a *stratum* of $GL(n,\mathbf{C})^p$: the stratum is defined by

1) the reducibility type of the group;

2) the Jordan normal forms of the small and big blocks of the matrices $M_1,\ldots,M_{p+1}$ and the ones of the matrices M_j themselves;

3) two groups whose matrices $M_1,\ldots,M_p$ are blocked as their reducibility type belong to the same stratum if and only if the corresponding M_j are conjugate to each other by matrices (in general, different for the different j) blocked as the reducibility type.

A stratum is called *irreducible* if its reducibility type is one big and at the same time small block.

A reducible stratum is called *special* if there exists a pair of small blocks of the same size, belonging to one and the same big block, such that the restrictions of the matrices M_j to them have the same Jordan normal form for all $j = 1, \ldots, p+1$.

Remark: Suppose that the definition of a stratum doesn't contain 3). Then some of the reducible strata defined in this way will turn out to be reducible analytic varieties (see the example below; note the double sense of 'reducible'). The good definition of a stratum is obtained when the strata defined above are decomposed into irreducible components if this is possible. After such a decomposition we obtain again a finite number of strata.

Example: Let the reducibility type be $\begin{pmatrix} P & Q \\ 0 & R \end{pmatrix}$, P, Q and R being 3×3.

Let $M_2, \ldots, M_{p+1}$ have distinct eigenvalues. Let $M_1 = \begin{pmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 0 & 0 & 0 & a \\ 0 & 0 & \lambda & b & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$.

For $a = 0, b \neq 0$ and for $a \neq 0, b = 0$ the Jordan normal forms of the $P-$ and $Q-$block of M_1 and of M_1 itself are the same (M_1 has one eigenvalue – λ – and three Jordan blocks, of sizes 3, 2 and 1 respectively). In the first case the dimension of the intersection of the subspace invariant for M_1 upon which M_1 acts as one Jordan block of size 3 with the subspace invariant for all operators M_j is equal to 1, in the second case it is equal to 2. It can be checked directly that the two matrices (corresponding to $(a,b) = (*,0)$ and $(a,b) = (0,*), * \neq 0$) aren't conjugate to each other by a matrix blocked in the same way.

Remark: The following example shows that the definition of a stratum of $GL(n, \mathbf{C})^p$ is still not good – there exist several connected components for irreducible strata in which every operator M_j, $j = 1, \ldots, p$ has one eigenvalue only. On the other hand-side, let there exist M_j with at least two different eigenvalues. Consider two systems belonging to the same stratum. One can deform continuously the sets of their eigenvalues, i.e. perform a homotopy from the first into the second set, keeping their product equal to 1 and their multiplicities unchanged, i.e. different (equal) eigenvalues remain such for every value of the homotopy parameter. Whether for any such homotopy there exists a homotopy of the monodromy group, irreducible for every value of the homotopy parameter – this is an open question.