Jean François Mattei
Marcel Nicolau

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EQUISINGULAR UNFOLDINGS OF FOLIATIONS BY CURVES

Jean François MATTEI and Marcel NICOLAU*

0. Introduction. Let F denote a holomorphic foliation by curves with isolated singularities on a complex surface M. The first author constructed in [M] a versal equisingular unfolding, parametrized by a smooth space of parameters K_i^{loc}, of the germ of the foliation F at one of its singular points q_i. The aim of this paper is to show the existence of a versal equisingular unfolding of the global foliation F when M is compact. In this case the parameter space K_e of the versal unfolding can be singular. The problem of finding conditions on F assuring the triviality of any unfolding has been considered by X. Gomez-Mont in [G-M].

An equisingular unfolding of F is an unfolding admiting a reduction of the singularities "with parameters". It is claimed in [M] that there is a one-to-one correspondence between equisingular unfoldings of F and locally trivial unfoldings (cf. Definition 1.6) of the reduction $\widetilde{F}$ of F preserving the divisor which comes from the singular points of F. So we are led to construct a versal locally trivial unfolding of a (possibly non saturated) foliation by curves. The construction of the versal space is carried out in the first two sections. The key point is the identification of locally trivial unfoldings with a certain type of deformations of the complex structure of the underlying manifold. Then we consider the relationship between the global versal space K_e and the local versal spaces K_i^{loc}. We show that under some cohomological assumptions K_e is smooth and naturally identified with the product $\prod K_i^{\mathrm{loc}}$. Finally we apply the above results to show that any equisingular unfolding of a germ of algebraic foliation is still algebraic.

1. Locally trivial unfoldings of foliations by curves.

Let M be a n-dimensional compact complex manifold and let TM be its holomorphic tangent bundle. Given a holomorphic vector bundle E over M we denote by $\mathcal{O}(E)$ the sheaf of germs of holomorphic sections of E. In

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particular $\mathcal{O}_M = \mathcal{O}(\mathbb{C})$ and $\Theta_M = \mathcal{O}(TM)$ are respectively the sheaves of germs of holomorphic functions and holomorphic vector fields on M.

By a (singular) holomorphic foliation on M we mean a locally free $\mathcal{O}_M$-submodule F of Θ_M which is closed under the Lie bracket of vector fields. The ***singular locus*** $S(F)$ of F is the analytic subset whose complementary $M - S(F)$ is the maximal open set on which F defines a foliation in the usual sense, i.e. without singularities. The foliation F is called ***saturated*** if one has

$$\Gamma(U, \Theta_M) \cap \Gamma(U - S(F), F) = \Gamma(U, F)$$

for any open subset $U \subset M$. It is well known that a foliation F is saturated if and only if $\operatorname{codim} S(F) \geq 2$. In fact it follows from Hartogs' extension theorem that if there is given an analytic subset $\Sigma \subset M$ of codimension greater than one and a non singular foliation F' on $M - \Sigma$ then there is a uniquely defined saturated foliation F on M which coincides with F' on $M - \Sigma$. In particular $S(F) \subset \Sigma$.

A ***foliation by curves*** is a locally free subsheaf F of Θ_M of rank one. Therefore a foliation by curves is determined by a pair (L, κ) where L is a line bundle over M and $\kappa : L \to TM$ is a non identically zero bundle morphism. The bundle morphism κ induces an injective morphism of sheaves $\mathcal{O}(L) \to \Theta_M$ and we identify $\mathcal{O}(L)$ with its image in Θ_M. Then $S(F)$ is the set of those points $z \in M$ for which $\kappa : L_z \to T_z M$ is the zero map.

In an equivalent way a holomorphic foliation by curves can be defined by a collection of local holomorphic vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ such that $\mathcal{U} = \{U_i\}$ is an open cover of M and $\xi_j = u_{ji} \cdot \xi_i$ on $U_i \cap U_j$ for suitable non vanishing holomorphic functions u_{ji}. Then L is the line bundle associated to the 1-cocycle $\{u_{ji}\}$. Moreover, if $\sigma_i : U_i \to L$ are non vanishing sections with $\sigma_j = u_{ji} \cdot \sigma_i$ then $\kappa : L \to TM$ is the bundle morphism determined by the condition $\kappa \circ \sigma_i = \xi_i$. The singular locus $S(F)$ is just the union $\bigcup \operatorname{Sing}(\xi_i)$ where $\operatorname{Sing}(\xi_i)$ is the subset of U_i where ξ_i vanishes.

To any foliation F there is naturally associated a saturated foliation ${}^sF$ which coincides with F outside $S(F)$. In the case of a foliation by curves ${}^sF$ can be described as follows. Assume that F is defined by local vector fields $\xi_i \in \Gamma(U_i, \Theta_M)$ where each U_i is holomorphically equivalent to an open polidisc Δ of $\mathbb{C}^n$. Let $z^1, \ldots, z^n$ be the coordinates on U_i induced by the identification $U_i \equiv \Delta$ and set $\xi_i = \sum \xi_i^\alpha \partial/\partial z^\alpha$. Let v_i be a m.c.d. of the functions $\xi_i^1, \ldots, \xi_i^n$ and define $\hat{\xi}_i = \xi_i / v_i$. Then $\operatorname{codim}(\operatorname{Sing}(\hat{\xi}_i)) \geq 2$. Since the foliation on $U_i \cap U_j$ defined by $\hat{\xi}_j$ is saturated there is a holomorphic function $\hat{u}_{ij}$ such that $\hat{\xi}_i = \hat{u}_{ij} \cdot \hat{\xi}_j$ on $U_i \cap U_j$. Furthermore the functions $\hat{u}_{ij}$ do not vanish. If not $\operatorname{Sing}(\hat{\xi}_j)$ would be of codimension one. Hence the local vector fields $\hat{\xi}_i$ define a saturated foliation ${}^sF$ called the ***saturation*** of F.

1.1. Example. Let a saturated foliation F and an analytic hypersurface D on M be given. One can find an open cover $\{U_i\}$ of M with the property that there exist $\xi_i \in \Gamma(U_i, \Theta_M)$ and $h_i \in \Gamma(U_i, \mathcal{O}_M)$ such that the collection $\{\xi_i\}$ defines F and $h_i = 0$ defines D on U_i (i.e. if $f \in \Gamma(W, \mathcal{O}_M)$ vanishes on D then $f = \lambda \cdot h_i$ on $W \cap U_i$). Set $\eta_i = h_i \cdot \xi_i$. The collection of vector fields $\{\eta_i\}$ defines a non saturated foliation by curves F^D with singular locus $S(F^D) = D \cup S(F)$ and whose saturation is just F. With more generality and for any given positive integer $k \in \mathbb{N}^*$ one can define the foliation $F^{k \cdot D}$ as the foliation by curves defined by the local vector fields $\eta_i^{(k)} = (h_i)^k \cdot \xi_i$. In section 3 we will consider locally trivial unfoldings of foliations by curves obtained in this way.

From now on F will be a fixed foliation by curves on a compact manifold M defined by a pair (L, κ). Let Ω be an open neighbourhood of 0 in $\mathbb{C}^m$. The product $\Omega \times M$ is endowed with a holomorphic foliation F_Ω of the same codimension as F obtained as the product of F by the foliation on Ω consisting of a single leaf; i.e. F_Ω is the foliation defined by the subsheaf $\mathrm{pr}_1^*\Theta_\Omega \oplus \mathrm{pr}_2^* F$ of $\Theta_{\Omega \times M}$. Here pr_1 and pr_2 denote respectively the natural projections from $\Omega \times M$ onto the first and second factors. We call F_Ω the *trivial unfolding* of F parametrized by Ω.

1.2. Definition. *Let W be an open subset of $\Omega \times M$ and let $\Psi : W \to \Psi(W) \subset \Omega \times M$ be an $\mathbb{C}$-analytic diffeomorphism over the identity of Ω such that the restriction of Ψ to $W \cap (\{0\} \times M)$ is the identity. We say that Ψ is a relative automorphism of the trivial unfolding F_Ω if there is a bundle morphism $\Psi^\sharp : (T\Omega \times L)|W \to (T\Omega \times L)|\Psi(W)$ over Ψ such that the diagram*

$$\begin{array}{ccc} (T\Omega \times L)|W & \xrightarrow{id \times \kappa} & (T\Omega \times TM)|W \\ \Psi^\sharp \downarrow & & \downarrow \Psi_* \\ (T\Omega \times L)|\Psi(W) & \xrightarrow{id \times \kappa} & (T\Omega \times TM)|\Psi(W) \end{array}$$

is commutative, where Ψ_* denotes the tangent map of Ψ.

1.3. Remark. Let Ψ be a local biholomorphism over the identity of Ω and inducing the identity on $\{0\} \times M$. Suppose that Ψ maps the singular locus $S(F_\Omega)$ of F_Ω identically into itself and preserves the foliation F_Ω outside $S(F_\Omega)$. If the foliation F is saturated then Ψ_* induces a bundle morphism $\Psi^\sharp$ fulfiling the conditions required in the above definition. This is no longer true in general if the foliation is not saturated and in this case Ψ need not to be an automorphism of the trivial unfolding. Nevertheless there is a particular type of non saturated foliations for which this property still holds. It is considered in Proposition 1.5.

Assume $\Psi : W \to \Psi(W)$ is a holomorphic diffeomorphism such that W, $\Psi(W)$ are subsets of $\Omega \times U$ where U is a coordinate open subset of M, with coordinates $z = (z^1, \ldots, z^n)$, on which F is defined by a holomorphic vector field ξ. Let $s = (s^1, \ldots, s^m)$ denote the linear coordinates on Ω. Then Ψ is a relative automorphism of F_Ω if and only if it is of the form $\Psi(s, z) = (s, \psi(s, z))$ where ψ is a holomorphic map such that $\psi(0, z) = z$ and fulfiling

$$(1) \qquad \psi_* \xi = (u \circ \Psi^{-1}) \cdot \xi,$$

$$(2) \qquad \psi_* \frac{\partial}{\partial s^\mu} = (v_\mu \circ \Psi^{-1}) \cdot \xi \qquad \text{for } \mu = 1, \ldots, m,$$

for suitable functions u, v_μ on W. For example, if $\varphi = \varphi(t, z)$ denotes the local flow of ξ and $\sigma = \sigma(s, z)$ is a holomorphic function with $\sigma(0, z) = 0$ then $\Psi(s, z) = (s, \varphi(\sigma(s, z), z))$ is a relative automorphism of the trivial unfolding. The following proposition states that any relative automorphism of F_Ω is locally of this form.

1.4. Proposition. *Let $\Psi(s, z) = (s, \psi(s, z))$ be a relative automorphism of F_Ω with domain $W \subset \Omega \times M$. Assume $U = W \cap (\{0\} \times M)$ is an open subset of M holomorphically equivalent to an open polidisc on which F is defined by a holomorphic vector field ξ. Let $\varphi = \varphi(t, z)$ be the local flow associated to ξ. Then there is a holomorphic function $\sigma = \sigma(s, z)$ defined in a neighbourhood W' of U in W with $\sigma(0, z) = 0$ and such that $\psi(s, z) = \varphi(\sigma(s, z), z)$ on W'.*

Proof. Because of (2) a function $\sigma = \sigma(s, z)$ fulfils the required conditions if and only if σ is a solution of the total differential equation (with parameter $z \in U$)

$$(3) \qquad \begin{cases} \dfrac{\partial \sigma}{\partial s^\mu}(s, z) = v_\mu(s, z) \\ \sigma(0, z) = 0. \end{cases}$$

The existence of such a function σ outside the singular locus $S(F_\Omega)$ of F_Ω can be easily seen by using local coordinates $w^1, \ldots, w^n$ such that $\xi = \partial / \partial w^n$. In particular the integrability condition of the equation (3), i.e.

$$(4) \qquad \frac{\partial v_\mu}{\partial s^\nu} = \frac{\partial v_\nu}{\partial s^\mu} \qquad \text{for } \mu, \nu = 1, \ldots, m,$$

is fulfiled outside $S(F_\Omega)$. But equalities (4) must then also be verified on the singular set by continuity. So the complex Frobenius theorem with holomorphic parameters applies showing that (3) has a unique solution defined in a neighbourhood of $\{0\} \times U$ in W. ∎

Let $F^{k \cdot D}$ denote a foliation of the type defined in Example 1.1. A relative automorphism of $(F^{k \cdot D})_\Omega$ is also a relative automorphism of F_Ω. The converse is not true in general. In the following proposition we consider a particular situation in which the converse still holds. It will be used in section 3.