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QUASI-REGULARITY PROPERTY FOR UNFOLDINGS OF HYPERBOLIC POLYCYCLES

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1. INTRODUCTION.

Let X be a real analytical vector field on $\mathbf{R}^2$. A polycycle Γ of X is an immersion of the circle, union of trajectories (Singular points and separatrices whose α and ω limits are contained in this set of singular points). Moreover one supposes that Γ is oriented by the flow of X and that a return map $P(x)$ along Γ is defined on some interval σ with one end point on Γ : σ is parametrized by analytical variable $x \in [0, x_1]$, $\{x = 0\} = \sigma \cap \Gamma = \{q\}$ and $P(x) : [0, x_0] \longrightarrow [0, x_1]$ for some $x_0 \in]0, x_1[$

We say that Γ is an hyperbolic polycycle if all the singular points in γ are hyperbolic saddle points. Let $\{p_1, \dots , p_k\}$ the set of these singular points listing in the way they are encountered when we describe Γ starting at q. We define the hyperbolicity ratio of p_i, $i = 1, \dots , k$ to be $r_i = \dfrac{\mu_i'}{\mu_i''}$ where $-\mu_i'$, μ_i'' are the eigenvalues at p_i ($\mu_i', \mu_i'' > 0$).

The Poincaré map $P(x)$ is analytic for $x > 0$, and extends continuously at 0 by $P(0) = 0$.

In 1985, Yu. Ilyashenko [I1] introduced a notion (the almost-regularity) similar to the following one up to a composition by the logarithm :

Definition. *Let $g(x) : [0, x_0] \longrightarrow \mathbf{R}$ a function, analytic for $x > 0$, and continuous at $x = 0$. One says that g is quasi-regular if:*

$\mathbf{Q}R_1$*) $g(x)$ has a formal expansion of Dulac type. This means that there exists a formal series:*

$$\hat{g}(x) = \sum_{i=0}^{\infty} x^{\lambda_i} P_i(ln x)$$

where λ_i is a strictly increasing sequence of positive real numbers $0 < \lambda_0 < \lambda_1 < \ldots$ tending to infinity and for each i, P_i is a polynomial, and $\hat{g}$ is a formal expansion of $g(x)$ in the following sense:

$$\forall n \geq 0 \qquad g(x) - \sum_{i=0}^{n} x^{\lambda_i} P_i(lnx) = o(x^{\lambda_n}).$$

QR_2) *Let $G(\xi) = g(e^{-\xi})$ for $\xi \in [\xi_0 = -logx_0, \infty[$.*

Then G has a bounded holomorphic extension in a domain $\Omega(C) \subset \mathbf{C}$ where $\Omega(C) = \{\zeta = \xi + i\eta \mid \xi^4 \geq C(1+\eta^2)\}$ for some $C > 0$.

In the same paper [I1], Ilyashenko proved that the shift map $\delta(x) = P(x) - x$ is quasi-regular. The property **Q**R_1 was already established by Dulac in [D].

As a consequence of the Phragmen-Lindelöf theorem (see [C]) a flat quasi-regular function ($g(x) = o(x^n), \forall n$) is necessarily equal to zero, and it follows from this that Γ cannot be accumulated by limit cycles of X (a limit cycle of X is an isolated periodic orbit).

This result was a first step in the solution of the "Dulac problem", for which one needs to look not only at hyperbolic polycycles but more generally at all elementary polycycles. As it is well known, this general solution ([EMMR], [E1], [E2], [I2], [I3]) involved more elaborated technics, and we limit ourselves to the hyperbolic polycycles in this paper.

Here we want to consider the unfoldings (X_λ, Γ) of a hyperbolic polycycle Γ, germs of finite parameter family (X_λ), with $X_0 = X$ defined by a representative family on $V \times W$ where V is a neighborhood of Γ and W neighborhood of 0 in the parameter space.

As it was shown in [R], it is useful to obtain quasi-regularity property for 1-parameter unfoldings, in order to study finite cyclicity for general unfoldings of hyperbolic polycycles. In the present paper, we extend to any 1-parameter unfoldings a result of [R], proved there for hyperbolic loops (singular cycles with just 1 singular point):

Theorem 1. *Let (X_ϵ, Γ) a 1-parameter analytic unfolding of an hyperbolic polycycle Γ for X_0 with k vertices. Let $P(x, \epsilon)$ the unfolding of the return map where x is some analytic parameter defined as above for X_0. Let $\delta(x, \epsilon) =$*

$P(x,\epsilon)-x$. Let $\widehat{\delta}(x,\epsilon)=\sum_{i=0}^{\infty}\delta_i(x)\epsilon^i$ (i.e: $\delta_i(x)=\frac{1}{i!}\frac{\partial^i\delta(x,0)}{\partial\epsilon^i}$) the formal expansion of δ in ϵ.

Then, there exists some $R>0$ (depending on $r_1(0),\dots,r_k(0)$) such that for $\forall i\in\mathbf{N}$, $x^{iR}\delta_i(x)$ is quasi-regular.

Remarks.

1) Given an unfolding X_ϵ and a transversal $\sigma\simeq[0,x_1]$ chosen as above for X_0, the return map $P(x,\lambda)$ is defined in a domain $D=\cup_{\epsilon\in W}[\alpha(\epsilon),x_1]$ where $\alpha(\epsilon)$ is a continuous function, such that $\alpha(0)=0$. So, given any $x\in]0,x_1]$, the return map $P(x,\lambda)$ is defined for x if $|\epsilon|$ is small enough. From this it follows that the functions $\delta_i(x)$ in the above theorem, are defined for $\forall x\in]0,x_1]$.

2) Theorem 1 extends Ilyashenko's one which corresponds to the quasi-regularity of $\delta_0(x)$.

The generalization brought by theorem 1 is useful to study unfolding of identical polycycles, i.e polycycles such that $\delta(x)=P(x)-x\equiv 0$. Suppose for instance that $\lambda=\epsilon\in\mathbf{R}$. Then, if (Γ,X_0) is an identical polycycle, one can write:

$$\delta(x,\epsilon)=\epsilon^n\bar{\delta}(x,\epsilon)$$

for some $n\geq 1$, with a function $\bar{\delta}(x,\epsilon)$ such that $\bar{\delta}(x,0)\not\equiv 0$. Then from theorem 1, we have that $\bar{\delta}(x,0)$ has a non-trivial Dulac expansion.

So, the equation for limit cycles $\{\delta(x,\epsilon)=0\}$, which is equivalent to $\{\bar{\delta}(x,\epsilon)=0\}$, has the same properties that in the non-identical case ($\delta(x,0)\not\equiv 0$).

This allows us to develop for some identical unfoldings a proof similar to the one for unfolding of non-identical polycycles. In [R] these ideas were applied to prove the finite cyclicity of any analytic unfolding of loops (Singular cycles with just one singular hyperbolic point). Here we extend it to some polycycles with 2 singular points:

Theorem 2. *Let (X_λ,Γ) an analytic unfolding of an hyperbolic 2-polycycle Γ (a polycycle with 2 singular points p_1,p_2). Let $r_1(\lambda)$, $r_2(\lambda)$ the λ-depending hyperbolicity ratio at p_1,p_2. Suppose that:*

1) For all λ, $r_1(\lambda)r_2(\lambda) \equiv 1$
2) at least one of the two saddle connexions remains unbroken (for all λ).
Then (X_λ, Γ) has a finite cyclicity.

Remark.

A part the conditions 1,2, no other conditions are imposed on (X_λ, Γ) and the polycycle Γ may be identical. The non-identical case was already worked out in a previous paper [El.M]. Moreover, if $r_1(0) = r_2(0)^{-1} \notin \mathbf{Q}$, a result of finite cyclicity was obtained in [M], without the conditions 1,2.

The conditions 1,2 in the theorem 2 may seem very restrictive. Netherthe-less the theorem has the following natural application to polynomial vector fields. Let P_{2p} be the family of all polynomial vector fields of some even degree $2p$, $p \geq 1$. It is easy to extend P_{2p} in an analytic family of vector fields on the sphere (X_λ). This family (X_λ) is equivalent to P_{2p} on the interior of a 2-disk D^2, whose boundary ∂D^2 corresponds to the "circle at infinity γ_∞". Singular points of (X_λ) appears at infinity in pairs of opposite points (p, q) and a consequence of the even degree is that the tangential eigenvalues at p, q are opposite and the same for the two radial eigenvalues. It follows that the product of the ratios of hyperbolicity at p and q is one. Then if for some value λ_0 (that we can suppose equal to 0), $X_{\lambda_0} = X_0$ has just a pair of singular points p, q on γ_∞ and if there exists a connection Γ_1 of p and q in $\text{int}(D^2)$, one can apply theorem 2 to the unfolding (X_λ, Γ) where Γ is one of the 2 polycycles containing Γ_1 and an arc Γ_2 of γ_∞ joigning p and q; we have $r_1(\lambda)r_2(\lambda) \equiv 1$ as noted above and the connection Γ_2 at infinity remains unbroken. This applies to the quadratic family $\mathcal{P}_2$ and allows to prove the finite cyclicity of some of the 121 possible cases of periodic limit sets listed for this family in [DRR] (cases labelled: H_1^1, H_2^1 in this article).

In the first paragraph, we prove the theorem 1. Of course, we hope that the quasi-regularity property proved here will have a more general application that the one given in theorem 2 and proved below in the second paragraph. In fact the proof uses the existence of a well ordered expansion for $\delta(x, \lambda)$ at any order of differentiability. This expansion was established for unfoldings like in theorem 2 in [El.M] and we recall it bellow. In this paper it was used to prove the finite cyclicity in the non-identical case. Here, we use it to reduce in some sense the general case to the non-identical case, by the method already described in the loop case in [R]. This is made in the second paragraph.

Firstly a natural ideal in the space of parameter functions germs, the