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A rigidity theorem for transverse dynamics of real analytic foliations of codimension one

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The purpose of this paper is to prove

Theorem 1. *Let $(M_i^n, \mathcal{F}_i), i = 1, 2,$ be real analytic and orientable foliations of n-manifolds of codimension 1 and $h : (M_1^n, \mathcal{F}_1) \to (M_2^n, \mathcal{F}_2)$ a foliation preserving homeomorphism. Assume that all leaves of $\mathcal{F}_1$ are dense and there exists a leaf of $\mathcal{F}_1$ with holonomy group $\neq 1, \mathbb{Z}$. Then h is transversely real analytic.*

This applies to prove the following topological rigidity of the Godbillon-Vey class of real analytic foliations of codimension one.

Corollary 2. *Let $(M_i, \mathcal{F}_i), h$ be as in Theorem 1. Then $h^*(\mathrm{GV}(\mathcal{F}_2) = \mathrm{GV}(\mathcal{F}_1)$ holds.*

Here $\mathrm{GV}(\mathcal{F}_i) \in H^3(M, \mathbb{R})$ denotes the Godbillon-Vey class of $\mathcal{F}_i$, which is represented by the 3-form $\alpha \wedge d\alpha$ with a C^∞ -1-form α on M such that $d\theta = \theta \wedge \alpha$ holds with a C^∞ -1-form θ defining $\mathcal{F}$. It is easy to see that the Godbillon-Vey class is invariant under C^2-diffeomorphisms. Ghys, Tsuboi [9] and Raby [18] proved the invariance under C^1-diffeomorphisms, while the invariance is known to fail in some C^0-cases (see [5,9,11]). (Corollary 2 seems to admit the various generalisations allowing the existence of compact leaves. But we will not touch on those generalisations. See also the papers [5,7].)

The proof of the C^1-invariance due to Ghys and Tsuboi is based on a certain rigidity for C^1-conjugacies of transverse dynamics of foliations along compact leaves as well as minimal exceptional leaves cutting Cantor sets on transverse sections. The proof of Theorem 1 is based on the topological rigidity theorem for pseudogroups of diffeomorphisms of $\mathbb{R}$ (Theorem 3(1)).

To state Theorem 3 we prepare some notions. Let Γ^ω_+ be the pseudogroup of real analytic and orientation preserving diffeomorphisms of open neighbourhoods of the line $\mathbb{R}$ respecting 0. We call a mapping $\phi : G \to \Gamma^\omega_+$ of a group G to the pseudogroup Γ^ω_+ a *morphism* if the set $\phi(G)_0$ of germs of $\phi(f), f \in G$ form a group and ϕ induces a group homomorphism of G to $\phi(G)_0$. Therefore $\phi(f) : U_{\phi(f)}, 0 \to \phi(f)(U_{\phi(f)}), 0$ is a real analytic diffeomorphism of open neighbourhoods of $0 \in \mathbb{R}$ for $f \in G$ representing the germ of $\phi(f)$. We call $\phi(G)_0$ the germ of $\phi(G)$ and say ϕ is *solvable* (respectively *commutative, etc*) if $\phi(G)_0$ is so. The *orbit* $\mathcal{O}(x)$ of an $x \in \mathbb{R}$ is the set of those x_l joined by a sequence $(x_0, x_1, ..., x_l)$ with $x = x_0, x_{i+1} = \phi(f_i)(x_i), x_i \in U_{\phi(f_i)}, i = 0, ..., l-1$ for arbitrary $l \geq 0$. The *basin* $B_{\phi(G)}$ of 0 is the set of those x for which the closure of the orbit $\mathcal{O}(x)$ contains 0. If $\phi(G)$ is non trivial, i.e. $\phi(f) \neq \mathrm{id}$ for an $f \in G$, $B_{\phi(G)}$ is an open neighbourhood of 0 [17]. Morphisms $\phi, \psi : G \to \Gamma^\omega_+$ are *topologically* (resp. C^r-) *conjugate* if there exists a homeomorphism (resp. C^r-diffeomorphism) $h : U, 0 \to h(U), 0$ of open neighbourhoods of 0 such that $U_{\phi(f)}, \phi(f)(U_{\phi(f)}) \subset U, U_{\psi(f)}, \psi(f)(U_{\psi(f)}) \subset h(U)$ and $h \circ \phi(f) = \psi(f) \circ h$ holds on $U_{\phi(f)}$ for all $f \in G$. We call h a *linking homeomorphism* (resp. *linking diffeomorphism*) and we denote $h : \phi \to \psi$.

Theorem 3 (The rigidity theorem for pseudogroups). *Let $\phi, \psi : G \to \Gamma^\omega_+$ be morphisms which are topologically conjugate with each other and $h : \phi \to \psi$ a linking homeomorphism.*

(1) If $\phi(G)_0, \psi(G)_0$ are not isomorphic to $\mathbb{Z}$ and non trivial, the restriction $h : B_{\phi(G)} - 0 \to B_{\psi(G)} - 0$ is a real analytic diffeomorphism.

(2) If $\phi(G)_0, \psi(G)_0$ are non commutative, h is unique and there exist even positive integers i, j such that $|h(\epsilon x^i)|^{1/j} : \tilde{B}^\epsilon_{\phi(G)} \to \tilde{B}^\epsilon_{\psi(G)}$ is a real analytic diffeomorphism for $\epsilon = \pm 1$. Here $\tilde{B}^\epsilon_{\phi(G)}$ is the set of those x such that $\epsilon x^i \in B_{\phi(G)}$ and $\tilde{B}^\epsilon_{\psi(G)}$ is the set of those x such that $x^j (resp. - x^j) \in B_{\psi(G)}$ if h maps $\mathbb{R}^\epsilon$ to $\mathbb{R}^+$ (resp. $\mathbb{R}^-$).

Now we apply the above rigidity theorem to the analytic action of the surface group on the circle S^1. Let Σ_g be the oriented closed surface of genus g and $\Gamma^g = \pi_1(\Sigma_g)$. For $r = 1, \ldots, \infty$ and ω, $\mathrm{Diff}^r_+(S^1)$ denotes the group of orientation preserving C^r-diffeomorphisms of the circle. The ***suspension*** M of a homomorphism $\phi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ is the quotient of $S^1 \times D^2$ by the product $\phi \times \Gamma$ with a discrete cocompact subgroup $\Gamma^g \simeq \Gamma \subset \mathrm{PSL}(2, \mathbb{R})$ acting freely on the interior of the Poincaré disc D^2. The second projection of $S^1 \times D^2$ induces the submersion of M onto $\Sigma_g = D^2/\Gamma$ with the fiber S^1. Since the action $\phi \times \Gamma$ respects the foliation of $S^1 \times D^2$ by the discs $x \times D^2, x \in S^1$, the suspension M is a foliated S^1-bundle of which the fibres are the quotients of the discs. In this way the topology of foliated S^1-bundles interchanges with that of the actions of Γ^g on S^1. The Euler number $\mathrm{eu}(\phi)$ of a homomorphism $\phi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ is defined to be that of the S^1-bundle associated to ϕ. The Milnor-Wood inequality [15,22] asserts

$$|eu(\phi)| \leq |\chi(\Sigma_g)| = 2g - 2.$$

The Euler number enjoys the following relations with the orbit structure:

(1) $\mathrm{eu}(\phi) = 0$ if there exists a finite orbit,

(2) If $\mathrm{eu}(\phi) \neq 0$, there exist a minimal set $\mathcal{M} \subset S^1$ of ϕ , an $x \in \mathcal{M}$ and an $f \in \mathrm{stab}(x)$ such that $\phi(f)|_{\mathcal{M}} \neq id$ [13], and if $r = \omega$ all orbits are dense [6] (see also [16]),

(3) If $|\mathrm{eu}(\phi)| = |\chi(\Sigma_g)|$ and $r \geq 2$, all orbits are dense [6],

where $\mathrm{stab}(x)$ denotes the stabiliser of x consisting of $f \in \Gamma^g$ with $\phi(f)(x) = x$. Homomorphisms $\phi, \psi : \Gamma^g \to \mathrm{Diff}^r_+(S^1)$ are ***C^s-conjugate*** if there exists a C^s-diffeomorphism h of S^1 such that $\psi(f) \circ h = h \circ \phi(f)$ holds for $f \in \Gamma^g$. We say ϕ, ψ are ***topologically conjugate*** if $s = 0$, ***semi conjugate*** if h is monotone map of degree one (possibly discontinuous). We call h a ***linking homeomorphism*** and denote $h : \phi \to \psi$. It is known that the Euler number (and the bounded Euler class) concentrate the homotopic property of the action, namely

Theorem(Ghys [3]). *ϕ, ψ are semi conjugate if and only if $\phi^*(\chi_{\mathbb{Z}}) = \psi^*(\chi_{\mathbb{Z}})$*

in the bounded cohomology group $H_b^2(\Gamma^g : \mathbb{Z})$, *where* $\chi_{\mathbb{Z}} \in H_b^2(\mathrm{Diff}_+^0(S^1) : \mathbb{Z}) = \mathbb{Z}$ *is the generator, the bounded Euler class.*

Theorem (Matsumoto [13]). *If* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi) = \pm\chi(\Sigma_g)$, ϕ, ψ *are semi conjugate, and if* $2 \leq r$, *they are topologically conjugate with each other, and in particular, conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$ *naturally acting on* S^1 *the boundary of the Poincaré disc.*

Theorem Ghys [8]. *If a homomorphism* $\phi : \Gamma^g \to \mathrm{Diff}_+^r(S^1)$ *attains the maximum of* $|\mathrm{eu}(\phi)|$ *and* $3 \leq r$, ϕ *is* C^r*-smoothly conjugate with a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{R})$.

In contrast to the above results, the properties of homomorphisms with $|\mathrm{eu}(\phi)| \lneqq |\chi(\Sigma_g)|$ are less known (see [16]). Applying Theorem 3 to the action of the stabiliser subgroup $\mathrm{stab}(x)$ on (S^1, x) for an $x \in S^1$, we obtain

Corollary 4. *Let* $\phi, \psi : \Gamma_g \to \mathrm{Diff}_+^\omega(S^1)$ *be homomorphisms with* $|\mathrm{eu}(\phi)|$, $|\mathrm{eu}(\psi)| \neq 0, |\chi(\Sigma_g)|$, *which are topologically conjugate, and* $h : \phi \to \psi$ *a linking homeomorphism. Assume that for an* $x \in S^1$, *the stabiliser subgroup* $\mathrm{stab}(x) \subset \Gamma_g$ *of* x *is not isomorphic to* $\mathbb{Z}$ *and non trivial. Then* h *is a real analytic diffeomorphism and orientation preserving or reversing respectively whether* $\mathrm{eu}(\phi) = \mathrm{eu}(\psi)$ *or* $\mathrm{eu}(\phi) = -\mathrm{eu}(\psi)$.

The statement remains valid for morphisms of groups G into $\mathrm{Diff}_+^\omega(S^1)$ replacing the condition on the Euler number by the existence of a dense orbit.

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2. Sequence geometry

In this paper $f^{(n)}$ denotes the n-fold iteration $f \circ \cdots \circ f$ of $f : U_f \to f(U_f)$ in Γ_+^ω. Let $\mathcal{X} = \{x_i\}, \mathcal{Y} = \{y_i\}, i = 1.2. \ldots$ be monotone sequences of positive numbers decreasing to 0. Define the *address function* $\mathrm{add}_{\mathcal{Y}}(x)$ of an $x > 0$ relative to $\mathcal{Y}$ to be the smallest integer i such that $y_i \leq x$. It is easy to see that $\mathrm{add}_{\mathcal{Y}}(x)$ is a decreasing function of x and $y_{\mathrm{add}_{\mathcal{Y}}(x)-1} > x \geq y_{\mathrm{add}_{\mathcal{Y}}(x)}$.