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C. Roche

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DENSITIES FOR CERTAIN LEAVES OF REAL ANALYTIC FOLIATIONS

C. ROCHE[1]

I. INTRODUCTION.

Let suppose an n dimensional real analytic manifold M be given. We will supose M to be paracompact connected and oriented. A real analytic $n-1$ *foliation with singularities* $\mathcal{F}$ on M is determined by giving an open covering (U_i) of M together with real analytic integrable 1-forms $\omega_i \in \Omega^1(U_i)$ such that on the overlapping charts, $U_i \cap U_j \neq \emptyset$, there exists a non vanishing function $g_{i,j} : U_i \cap U_j \to \mathbf{R}^*$ such that $\omega_i = g_{i,j}\omega_j$. Leaves of $\mathcal{F}$ on U_i are unions of the integral manifolds of the pfaffian equation $\omega_i = 0$.

The singular set of the foliation Sing($\mathcal{F}$) is the analytic subspace of M defined by the annulation of the forms ω_i. In local coordinates of M, each ω_i can be written as

$$\omega_i(x) = \sum_{l=1}^{n} a_l^i(x) dx^l$$

and locally Sing($\mathcal{F}$) is determined by the equations

$$a_1^i(x) = 0, \ldots, a_n^i(x) = 0 \qquad x \in U_i.$$

The hypothesis that the $g_{i,j}$ be non vanishing allowes to suppose that the singular set is of codimension at least 2. Such $\mathcal{F}$ defines on $M \setminus \mathrm{Sing}(\mathcal{F})$ an $n-1$ dimensional analytic foliation: $\mathcal{F}_{reg}$. Leaves of $\mathcal{F}_{reg}$ are called regular leaves of $\mathcal{F}$.

Morover if we suppose $\mathcal{F}$ to be transversally orientable, as will be done in this paper, Theorem A and B of Cartan in the real case [3] show that we can glue the 1-forms in order to suppose that the foliation $\mathcal{F}$ is given by a globally defined real analytic differential form ω, that is $\omega_i = \omega_{|U_i}$.

Consider now a union Γ of regular leaves of such a foliation $\mathcal{F}$, Γ is an immersed $n-1$ real analytic submanifold of M. Γ is called a separating solution

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by Khovanskii if there are two disjoint open sets, L_1 ans L_2 of M such that $M\setminus$ Sing$(\mathcal{F}) \setminus\Gamma = L_1 \cup L_2$, $\Gamma = \bar{L_1} \setminus L_1\setminus$Sing$(\mathcal{F})$ and finally ω points inside L_1 all along Γ.

In [17] we generalize this notion introducing Rollian pfaffian hypersurfaces. A regular leaf V of $\mathcal{F}$ is so called if for each analytic path $\gamma : [0,1] \to M$ intersecting the set V twice, say $\gamma(0) \in V$ and $\gamma(1) \in V$ there is an intermediate point, say $\gamma(t)$, $t \in [0,1]$ where the path is tangent to $\mathcal{F}$. At this point, if $\mathcal{F}$ is determined by the pfaff equation $\omega = 0$

$$\omega(\gamma(t)).\gamma'(t) = 0.$$

Such a Rollian pfaffian hypersurface (Rollian leaf or Rollian ph for short) will be denoted $\{V, \mathcal{F}, M\}$ to emphasize the pfaffian equation verified by V.

Khovanskii's Rolle theorem asserts that every separating solution of $\omega = 0$ is a union of Rollian ph. Separating solutions are not easy to find but, as it was shown in [17], an argument of Haefliger proves that if $M\setminus$Sing$(\mathcal{F})$ is simply connected, each regular leaf of $\mathcal{F}$ is a Rollian ph.

In [17] we used this generalisation to prove the following general finiteness theorem.

Theorem on uniform finiteness. *Let $\mathcal{F}_1, \dots, \mathcal{F}_q$ be transversally oriented singular foliations on M. If X is a semianalytic subset of M for each compact set K of M there is a constant $b \in \mathbf{R}$ such that for any set of Rollian pfaffian hypersurfaces $\{V_i, \mathcal{F}_i, M\}, i = 1, \dots, q$ the number of connected components of $X \cap V_1 \cap \dots \cap V_q$ meeting K is bounded by b.*

A carefull reading of the proof of this theorem in [17] shows that a separating manifold is in fact a locally finite union of Rollian ph as was shown by Khovanskii [5].

As an easy consequence of this result we can mention that a Rollian ph $\{V, \mathcal{F}, M\}$ is a real analytic submanifold of M closed in $M\setminus$Sing$(\mathcal{F})$.

In developping the ideas sketched in Khovanskii's work [5] [6] in joint work with R. Moussu, J.-M. Lion and J.-Ph. Rolin (started in [16]) we tried to consider Rollian ph just as building blocks for a theory similar to that of semianalytic sets. By different methods the same goal is pursued by Tougeron [19]. This idea leads to the problem of the behaviour of the boundary of a Rollian ph. At present time it is not known if the closure of a Rollian ph $\{V, \mathcal{F}, M\}$, $\bar{V}$ can be stratified with some regularity condition. In a forthcomming paper of F. Cano, J.-M. Lion and R. Moussu an important result on the regularity of the boundary $\bar{V} \setminus V$ of such a Rollian ph will be described. [2].

The study of the boundary of a sole Rollian ph $\{V, \mathcal{F}, M\}$ is most usefull for further research if we describe the structure of the boundary of an intersection $X \cap V$ where X is a semianalytic subset of M. If X is open connected and relatively compact in M, $X \cap V$ is a finite union of leaves of the restricted foliation $\mathcal{F}_{|X}$ each of them is a Rollian ph in X. The behabiour of V at the ends of M is so permited in the case the foliation can be regularly continued.

Let's define a pfaffian subset of M as a finite intersection $W = X \cap V_1 \cap \cdots \cap V_q$ where X is any semianalytic subset of M and the $V_i's$ are Rollian ph of foliations $\mathcal{F}_i$.

The following properties are known for the set $\partial W = \bar{W} \setminus W$. See [8][10].

Theorem on finiteness of the boundary. *The set ∂W is locally arc connected. Morover if $B_a(\rho)$ is the euclidean open ball of center a and radius ρ for $a \in \bar{W}$ the number of connected components of $\partial W \cap B_a(\rho)$ can be bounded by a constant depending only on the foliations $\mathcal{F}_i$ but not on the particular Rollian ph chosen.*

Let $C_y(A)$ be the tangent cone of $A \subset M$ at $y \in M$.

Curve selection lemma. *Let $a \in \partial W$, $u \in C_a(W)$, with $||u|| = 1$ be given, there is a semianalytic subset Y of M such that $W \cap Y$ is a union of paths $\gamma_i((0,1))$ one of them, say γ_0, can be extended in a C^1 way at 0 by $\gamma_0(0) = a$ and $\gamma_0'(0) = u$.*

These curves are pfaffian curves.

In this paper we show that Rollian ph have local volume properties similar to those of semianalytic and subanalytic sets.

A subset Y of $\mathbf{R}^n$ has a k-dimensional density at $y \in \mathbf{R}^n$ if the k-dimensional volume of $B_y(\epsilon) \cap Y$, $vol_k(B_y(\epsilon) \cap Y)$ is finite for small enough $\epsilon > 0$ and the following limit exists

$$\Theta_k(Y, y) = \lim_{\epsilon \to 0+} \frac{vol_k(B_y(\epsilon) \cap Y)}{\epsilon^k}.$$

This quantity is called density of Y at y. If these conditions are not fullfilled we can always consider the corresponding superior limit and inferior limit, which are denoted by $\bar{\Theta}_k(Y, y)$ and $\underline{\Theta}_k(Y, y) \in \bar{\mathbf{R}}_+$ respectively.

In a recent paper [7] Kurdyka and Raby show that subanalytic subsets have a density at every point. Our result is similar, but restricted to the case of Rollian ph as we cannot, at present time, obtain a general decomposition into graphs theorem for pfaffian sets.

Precisely, let M be an open semianalytic subset of $\mathbf{R}^n$

Theorem 1. *Let $\{V, \mathcal{F}, M\}$ be a Rollian pfaffian hypersurface then V has a density at each point of $\bar{V}$.*

The proof of this result uses the same idea of Kurdyka ans Raby and needs a new result on decomposition of Rollian ph into graphs. This decomposition gives a precision to a similar result of Lion [8], [9] and is obtained in a more elementary way. Namely

Proposition 1. *Let ω be an integrable real analytic 1-form, in a neighborhood of $0 \in \mathbf{R}^n$ and a small enough $\epsilon > 0$ be given. Then there is a finite number of hyperplans (H_i) and a subanalytic stratification $\mathcal{N}$ of a ball $B_0(\rho)$ such that: if $\{V, \omega, B_0(\rho)\}$ is a Rollian ph and $N \in \mathcal{N}$ then either*
$V \cap N$ is included in a smooth submanifold of dimension less than $n-1$,
or $V \cap N \subset H_i \oplus H_i^{\perp} \subset \mathbf{R}^n$ is the graph of a locally ϵ–lipschitzian analytic function on an open subset of H_i.

That is, up to a smaller dimensional set, each Rollian ph is a graph of an analytic funtion. This function can be supposed to have a very small derivative.

It is known that strong regularity conditions for stratified objects doesn't imply the existence of densities. Theorem 1 gives an interesting information on the good behaviour of the boundary of a Rollian ph even in case a theorem of regular stratification happens to be obtained.

The generalisation of theorem 1 to all pfaffian sets would be not difficult provided a result similar to Proposition 1 for several pfaffian equations can be proved.

II. Tangents to semianalytic sets and pfaffian equations.

Here we discuss a general stratification procedure preparing a graph decomposition of Rollian ph. In the first two paragraphs the discusion is fairly general and we restrict to the case of a single pfaffian equation in the third paragraph in order to get the proof of Proposition 1. We will use freely the theory of semianalytic sets [1] and stratifications [15]. A stratification is said to be adapted to a set if this set is a union of strata.

The proofs being local we will suppose from now on that M is an open semianalytic subset of $\mathbf{R}^n$.

1.Strongly analytic submanifolds. A subset X of M is a strongly analytic submanifold of M if it is semianalytic in M and a submanifold of M. That is locally at each point *of* X , X is given by the level set of an analytic submersion