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VANISHING HOLONOMY AND MONODROMY OF CERTAIN CENTRES AND FOCI

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Introduction

Let $\omega(x,y) = A(x,y)dx + B(x,y)dy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at the origin: $A(0,0) = B(0,0) = 0$, $dim_{\mathbf{R}} \frac{\mathbf{R}\{x,y\}}{(A,B)} < +\infty$.

The singularity $\omega = 0$ is called *monodromic* if there are not separatrices at 0. In this case, given a germ of an analytic embedding $(\mathbf{R}^+,0) \xhookrightarrow{\tau} (\mathbf{R}^2,0)$ transverse to ω outside 0, it is possible to define a *monodromy map* $P_{\omega,\tau}$: $(\mathbf{R}^+,0) \to (\mathbf{R}^+,0)$, following clockwise the solutions of $\omega = 0$; $P_{\omega,\tau}$ is a germ of homeomorphism of $(\mathbf{R}^+,0)$ analytic outside 0. If $P_{\omega,\tau} = id$ then $\omega = 0$ is called *centre*. Otherwise $P_{\omega,\tau}$ is a contraction or an expansion (by the results of Écalle, Il'yashenko, Martinet, Moussu, Ramis... on "Dulac conjecture") and $\omega = 0$ is called *focus*.

The simplest monodromic singularities are those for which the linear part M_ω of the dual vector field $v(x,y) = B(x,y)\frac{\partial}{\partial x} - A(x,y)\frac{\partial}{\partial y}$ is nondegenerate, i.e. invertible. We distinguish two situations:

i) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with real part different from zero. Then $\omega = 0$ is a focus and it is analytically equivalent to $\omega_{lin} = 0$, where ω_{lin} denotes the linear part of ω (Poincarè's linearization theorem).

ii) the eigenvalues λ, μ of M_ω are complex conjugate, non real, with zero real part. Then if $\omega = 0$ is a centre there exists an analytic first integral (Lyapunov-Poincarè theorem, see [Mou1] and references therein) and $\omega = 0$ is analytically equivalent to $xdx + ydy = 0$. If $\omega = 0$ is a focus the analytic classification is a difficult problem, which requires the theory of Écalle-Martinet-Ramis-Voronin to pass from the formal classification to the analytic one ([M-R]). The monodromy is an analytic diffeomorphism tangent to the identity, and two such equations are analytically equivalent if and only if their monodromies are ([M-R]).

In this paper we shall study the simplest degenerate monodromic singularities, i.e. those with $\lambda = \mu = 0$, $\omega_{lin} \neq 0$, and with "generic" higher order terms. Modulo a change of coordinates ([Mou2]), we may work in the following class.

Definition. Let $\omega = Adx + Bdy = 0$ be the germ of an analytic differential equation on $\mathbf{R}^2$, with an algebraically isolated singularity at 0. This singularity is called *monodromic semidegenerate* if the first nonzero quasihomogeneous jet of type (1,2) of ω is

$$\omega_0(x, y) = x^3dx + (y + ax^2)dy$$

with $a^2 < 2$. Notation: $\omega \in MSD(a)$.

We will denote by P_ω the monodromy map of $\omega \in MSD(a)$ corresponding to the embedding $(\mathbf{R}^+, 0) \hookrightarrow (\mathbf{R}^2, 0)$, $t \mapsto (t, 0)$. P_ω is a germ of analytic diffeomorphism tangent to the identity ([Mou2]), and we may consider P_ω as the restriction to $\mathbf{R}^+$ of a germ of biholomorphism of $(\mathbf{C}, 0)$, tangent to the identity, again denoted by P_ω.

Let $\omega \in MSD(a)$ and let Ω be the germ of holomorphic 1-form on $\mathbf{C}^2$ obtained by complexification of ω. Using a resolution of the singularity we may define as in [Mou3] and [C-M] the *vanishing holonomy* of Ω: it is a subgroup $H(\Omega) \subset Bh(\mathbf{C},0) = \{$ group of germs of biholomorphisms of $(\mathbf{C},0)\}$, generated by $f, g \in Bh(\mathbf{C},0)$ satisfying the relation $(f \circ g)^2 = id$.

Our result is a computation of P_ω in terms of $H(\Omega)$. A similar result was remarked by Moussu in the (simpler) case of nondegenerate monodromic singularities ([Mou1]).

Theorem. *Let $\omega = 0$ be monodromic semidegenerate, then*

$$P_\omega = [f, g]$$

In particular, $H(\Omega)$ is abelian if and only if $\omega = 0$ is a centre. This means, by [C-M], that a nontrivial space of "formal-analytic moduli" can appear only if $\omega = 0$ is a centre (and $a = 0$, see below): for the foci, formal equivalence $\Rightarrow$ analytic equivalence. Hence our situation is very different from the situation of equations of the type $xdx + ydy + ... = 0$, where the difficult case is the case of foci whereas all the centres are analytically equivalent (here the vanishing holonomy is always abelian, generated by a single $f \in Bh(\mathbf{C},0)$, and the monodromy is given by f^2, see [Mou1]). On the other hand, it is no more true that the monodromy characterizes the equation: it may happen that $\omega_1, \omega_2 \in MSD(a)$ have the same monodromy without being analytically equivalent.

A consequence of the above relation between monodromy and vanishing holonomy is the following normal form theorem for centres, based again on the results of [C-M]. Let us before remark that $\omega_0(x,y) = x^3dx + (y + ax^2)dy = 0$ is a centre for any $a \in \mathbf{R}$ (but a first integral exists if and only if $a = 0$).

Corollary 1. *Let $\omega \in MSD(a)$ be a centre and let $a \neq 0$, then the germ $\omega = 0$ is analytically equivalent to $\omega_0 = 0$.*

We don't know a similar explicit and "simple" (polynomial?) normal form for foci, even in the case $a \neq 0$; but the triviality of the space of formal-analytic moduli seems here a useful tool. The classification of centres with $a = 0$ requires arguments of the type Écalle - Martinet - Ramis - Voronin (cfr. [C-M]).

As another corollary of the above theorem we give a positive answer to a question posed by Moussu in [Mou2].

Corollary 2. *Let $\omega = 0$ be a monodromic semidegenerate centre, then there exists a nontrivial analytic involution $I : (\mathbf{R}^2, 0) \to (\mathbf{R}^2, 0)$ which preserves the solutions of $\omega = 0$: $I^*(\omega) \wedge \omega = 0$.*

The above computation may be generalized to the case of germs ω whose first nonzero quasihomogeneous jet of type $(1, n)$ is

$$\omega_0(x, y) = x^{2n-1}dx + (y + ax^n)dy$$

with $a^2 < \frac{1}{4}n$ ([Mou2]). The vanishing holonomy $H(\Omega)$ for these germs is generated by $f, g \in Bh(\mathbf{C}, 0)$ satisfying $(f \circ g)^n = id$ ([C-M]). But now, if $n \geq 3$, the relation between commutativity of $H(\Omega)$ and triviality of P_ω becomes more complicated; in particular, it is no more true that there is equivalence between "$H(\Omega)$ abelian" and "$P_\omega = id$".

The computation of P_ω in terms of $H(\Omega)$ for $n \geq 3$ is straightforward, once one has understood the case $n = 2$. Hence, for sake of simplicity and clarity, we have choose to limit ourselves to the semidegenerate monodromic singularities.

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Resolution of singularities and vanishing holonomy

Let $\omega \in MSD(a)$ and let Ω be its complexification. We recall the desingularization of Ω and the construction of $H(\Omega)$ ([C-M], [Mou3]).