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THE BOUNDARY OF THE MANDELBROT SET HAS HAUSDORFF DIMENSION TWO

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INTRODUCTION

Let $P_c(z) = z^2 + c$ $(z, c \in \mathbf{C})$. The Julia set J_c of P_c and the Mandelbrot set are defined by

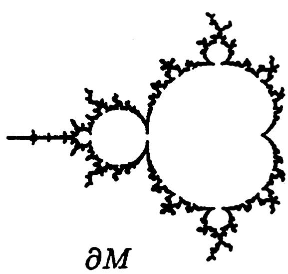
$$J_c = \text{the closure of } \{ \text{ repelling periodic points of } P_c\},$$
$$M = \{c \in \mathbf{C} \mid J_c \text{ is connected}\} = \{c \in \mathbf{C} \mid \{P_c^n(0)\}_{n=1}^{\infty} \text{ is bounded}\}.$$

In [**Sh**], we proved the following theorems concerning the Hausdorff dimension (denoted by $H-\dim(\,\cdot\,)$) of these sets:

Theorem A. *(conjectured by Mandelbrot [**Ma**] and others)*

$$H - \dim \partial M = 2.$$

Moreover for any open set U which intersects ∂M, we have $H-\dim(\partial M \cap U) = 2$.



∂M

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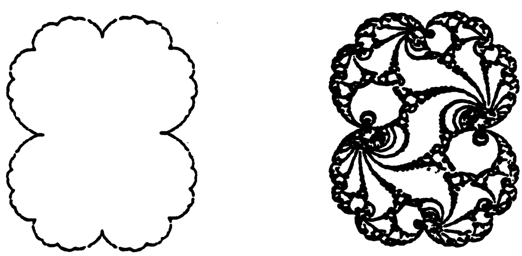
Theorem B. *For a generic $c \in \partial M$,*

$$\text{H-dim}\, J_c = 2.$$

In other words, there exists a residual (hence dense) subset $\mathcal{R}$ of ∂M such that if $c \in \mathcal{R}$, then H-dim $J_c = 2$.



J_c for $c = \frac{1}{4}$ and for $c = 0.25393 + 0.00048i$

In order to explain the idea of the proof, we need to introduce the following.

Definition. A closed subset X of $\overline{\mathbb{C}}$ is called a *hyperbolic subset* for a rational map f, if and only if $f(X) \subset X$ and f is expanding on X (i.e. $\|(f^n)'\| \geq C\kappa^n$ on X $(n \geq 0)$ for some $C > 0$ and $\kappa > 1$, where $\|\cdot\|$ denotes the norm of the derivative with respect to the spherical metric of $\overline{\mathbb{C}}$). The *hyperbolic dimension* of f is the supremum of H-dim(X) over all hyperbolic subset X for f.

Theorem A was an immediate consequence of the following claims $(*)$ and $(**)$ (Corollary 3 (i) and (ii) in [Sh]).

$(*)$ If U is an open set containing $c \in \partial M$, then

$$\text{H-dim}(\partial M \cap U) \geq \text{hyp-dim}(P_c).$$

$(**)$ For $c \in \partial M$, there exists a sequence $\{c_n\}$ in ∂M such that $c_n \to c$ and hyp-dim$(P_{c_n}) \to 2$, as $n \to \infty$.

Theorem B also follows from $(**)$, as we will see in §1.

In this paper, we give a brief explanation on the hyperbolic dimension and on the proof of $(*)$ (in §1), and present a different point of view for the proof of (**) after McMullen [Mc2], although the essential idea came from [Sh]. The claim (**) is reduced to the main theorem in §2, which is reduced to Proposition 5.1 in §5.

The main Theorem has a meaning that the "secondary bifurcation" of the parabolic fixed point leads to high hyperbolic dimension (i.e. close to 2). The proof presented in this paper is based on the notion of geometric limit, which was proposed by McMullen [Mc2]. By a geometric limit, we mean a limit of iterates $f_n^{k_n}$ in some region (usually only in a proper subregion of $\overline{\mathbb{C}}$), where $k_n \in \mathbb{N}$, $k_n \to \infty$.

In §3, we construct a geometric limit of rational maps related to a parabolic fixed point. This case was studied earlier by Lavours [La]. In §4, we construct a second geometric limit, which corresponds to the secondary bifurcation. These limits are used in §5, to construct "good" hyperbolic subsets with high Hausdorff dimension, which leads to the proof of the main theorem. Here the use of second geometric limits is crucial.

I would like to thank specially Curt McMullen for inspiring discussions on this subject. The above pictures are made by using J. Milnor's program.

§1. Hyperbolic subsets and hyperbolic dimension

This section is devoted to the topics related to hyperbolic subsets and hyperbolic dimension. We state some properties the hyperbolic dimension, prove that (**) implies Theorem B, state the outline of the proof of (*), and explain a specific construction of a hyperbolic subset.

Properties of hyperbolic dimension. (see [Sh] §2 for the proof.)

Let X be a hyperbolic subset for a rational map f.

(1.1) *X is a subset of the Julia set $J(f)$ of f. Hence* H-dim $J(f) \geq$ hyp-dim(f).

(1.2) *X is "stable" under a perturbation, i.e. a nearby rational map f_1 has a hyperbolic subset X_1 such that (X_1, f_1) is conjugated to (X, f) by a Hölder homeomorphism with Hölder exponent close to* 1.

(1.3) $f \mapsto$ hyp-dim(f) *is lower semi-continuous, or equivalently, for any number a, the set* $\{f|$ hyp-dim$(f) > a\}$ *is open.*

Now we can prove that (**) implies Theorem B.

Proof of Theorem B. Let $\mathcal{R}_n = \{c \in \partial M |\ \text{hyp-dim}(P_c) > 2 - \frac{1}{n}\}$ $(n = 1, 2 \cdots)$. Then $\mathcal{R} = \bigcap_{n>0} \mathcal{R}_n = \{c \in \partial M |\ \text{hyp-dim}(P_c) = 2\ \} \subset \{c \in \partial M |\ \text{H-dim}\ J(P_c) = 2\ \}$. By Property (1.3), $\mathcal{R}_n$ are open in ∂M. Moreover $\mathcal{R}_n$ is dense in ∂M by (**). Hence $\mathcal{R}$ is residual (i.e. contains the intersection of a countable collection of open dense subsets of ∂M.) Such an $\mathcal{R}$ is dense in ∂M by Baire's theorem. □

Remark 1.4. It follows from (1.3) that if $\text{hyp-dim}(P_{c_0}) = 2$ then both $c \mapsto \text{hyp-dim}(P_c)$ and $c \mapsto \text{H-dim}(J_c)$ are continuous at $c = c_0$. On the other hand, (**) implies that if $c_0 \in \partial M$ and $\text{hyp-dim}(P_{c_0}) \neq 2$ (resp. $\text{H-dim}(J_{c_o}) \neq 2$) then $c \mapsto \text{hyp-dim}(P_c)$ (resp. $c \mapsto \text{H-dim}(J_c)$) is not continuous at $c = c_0$. Note that P_c is J-stable in $\mathbb{C} - \partial M = (\mathbb{C} - M) \cup (\text{interior} M)$, hence $c \mapsto \text{hyp-dim}(P_c)$ and $c \mapsto \text{H-dim}(J_c)$ are continuous.

In fact, if X is a hyperbolic subset for f and a homeomorphism conjugates f to f_1 in a neighbourhood of the Julia sets, then $h(X)$ is a hyperbolic subset for f_1.

Outline of the proof of (*). The main idea is "the similarity between the Mandelbrot set and Julia sets" (cf. [T]). See [Sh] §3 for the complete proof.

Let $c_0 \in \partial M$. For any $\varepsilon > 0$, there exists a hyperbolic subset X for P_{c_0} such that $\text{H-dim}\, X > \text{hyp-dim}(P_{c_0}) - \varepsilon$. Let us take a point $z \in X$ such that X intersected with any neighborhood of z has the same Hausdorff dimension as entire X. By Property (1.2), for c in a neighborhood U of c_0, P_c has a hyperbolic subset X_c ($X_{c_0} = X$), which moves continuously with c.

On the other hand, the family $\{P_c^n(0)\}_{n=1}^{\infty}$ (as functions of c) is not normal in any neighborhood of c_0. In fact, $\{P_{c_0}^n(0)\}$ is bounded, but for arbitarily close $c \in \mathbb{C} - M$, $P_c^n(0) \to \infty$ ($n \to \infty$). Then one can show that for z chosen above, there are a parameter c_1 arbitrarily close to c_0 and a positive integer m such that $P_{c_1}^m(0)$ belongs to X_{c_1} and corresponds to z in X.

Now consider $M_1 = \{c \in U |\ P_c^m(0) \in X_c\}$. One can show that $M_1 \subset \partial M$. For $c \in M_1$, let h(c) be the point in X corresponding to $P_c^m(0)$ in X_c. Consider the case where $P_c^m(0)$ moves "transversally" relative to X_c. Then one can show that h is a homeomorphism from a neighborhood of c_1 in M_1 to a neighborhood of z in X. Moreover it is Hölder with exponent close to 1, for c_1 near c_0. (To see these facts, pretend that X_c did not move at all. Then h would give a local diffeomorphism near c_1.)

Therefore M_1 has Hausdorff dimension close to $\text{H-dim}\, X$ or higher. So we can prove the assertion by letting $c_1 \to c$ and $\varepsilon \to 0$. □

In §5, we will construct a special kind of hyperbolic sets as follows.

Lemma 1.5. *Suppose that U is a simply connected open set in $\mathbb{C}$; $U_1, \ldots, U_N$ are disjoint open subsets of U with $\overline{U}_i \subset U$; $n_1, \ldots, n_N$ are positive integers such that f^{n_i} maps U_i onto U bijectively ($i = 1, \ldots, N$).*

Then there exists a Cantor set X_0 generated by $\tau_i \equiv (f^{n_i}|_{U_i})^{-1} : U \to U_i$ ($i = 1, \ldots, N$), that is, X_0 is the minimal non-empty closed set satisfying

$$X_0 = \tau_1(X_0) \cup \cdots \cup \tau_N(X_0),$$