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Perturbations of Critical Fixed Points of Analytic Maps

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This paper has three parts. The first is concerned with an inequality relating the positions of the fixed point, the critical point, and the critical value obtained by a perturbation of an analytic function which has a critical point which is simultaneously a fixed point. The result is local in nature in that the analytic function need only be defined in a neighborhood of the fixed critical point. However, the motivation for this problem comes from a conjecture of Smale about polynomials, [4].

The conjecture states that, given a complex polynomial f of degree d and a non-critical point z, that for some critical point θ

$$(1) \qquad |f(z) - f(\theta)| \,/\, |(z - \theta)f'(z)| < 1.$$

Let $S(f, z, \theta)$ denote the left hand side of (1). For the polynomial $f(z) = z^d + (d/(d-1))z$, and the point $z = 0$, for any choice of critical point θ, $S(f, 0, \theta) = (d-1)/d$. This is possibly the worst case for the conjecture, which would mean that $(d-1)/d$ could replace 1 in (1). In [5], we showed that $(d-1)/d$ is an upper bound for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$ and $z = 0$. We also conjectured that the mean value conjecture might be strengthened to state that for any f of degree d, and any z, for some θ

$$(2) \qquad |S(f, z, \theta) - 1/2| \leq 1/2 - 1/d.$$

In the first part of this paper we will show that (2) is true for any sufficiently small perturbation of $f(z) = z^d + (d/(d-1))z$, $z = 0$.

In the second part of this paper we will consider a topological version of the mean value conjecture. That is, we will consider only critical points θ for which $f(\theta)$ is on the boundary of the largest disk centered at $f(z)$ on which a branch of f^{-1} can be defined. For these critical points, by applying the Koebe 1/4 theorem to f^{-1}, it was shown in [4] that $S(f, z, \theta) \leq 4$. The topological version of the mean value conjecture raises the question of whether there is a better version of the Koebe 1/4 theorem for inverse branches of polynomials.

We will consider degrees $d = 3, 4$ where (2) is known to be true, [5]. We will show that the topological version of (2) is true for $d = 3$ and is false for $d = 4$. Whether the topological version of (1) is true is not known.

In the final section we will describe a quasi-conformal model for polynomials which will allow us to verify (2) for roots of polynomials whose critical values have norms that increase fast enough.

§1. Let $f(z)$ be a complex analytic function defined in a neighborhood of the point θ. Suppose that $f(\theta) = \theta$ and that $f'(\theta) = 0$ and $f''(\theta) \neq 0$. Let p be another point different than θ, not necessarily in the domain of f. Since θ is a fixed point of f, $|\theta - p| = |f(\theta) - p|$. Let h denote a complex analytic function which is a peturbation of f in a neighborhood of θ. For any sufficiently small perturbation h of f there will be a fixed point q, $(h(q) = q)$, and a critical point σ, $(h'(\sigma) = 0)$, near θ. The following theorem gives a sufficient condition on perturbations of f so that $|\sigma - p| \geq |h(\sigma) - p|$.

Theorem 1. *Suppose that* $\mathbf{R}((\theta - p)(f''(\theta)) < 0$. *Suppose that* $|\sigma - p| > |q - p|$. *Then if h is sufficiently near f,* $|h(\sigma) - p| \leq |\sigma - p|$.

Proof. Without loss of generality, for the purposes of this proof, we can assume that $p = 0$. This follows because conjugating f by a translation does not change the second derivative, and translations preserve lengths. Since $p = 0$ and $p \neq \theta$, we can define an analytic function g in a neighborhood of θ by the formula $f(z) = zg(z)$. We are interested in the level curves of g. That is the curves defined by $|g(z)|$ =constant.

Lemma 1. *Let $f(z) = zg(z)$ and suppose $f'(\theta) = 0$ and $g'(\theta) \neq 0$. Then, at θ, the tangent vector $-\theta$ is orthogonal to the level curve of g.*

Proof. Let $w = g(z)$. The level curves of g are the preimages of circles in the w plane centered at 0. The radial vector field $V(w) = w$ pulls back by g to the vector field $g(z)/g'(z)$. Since g is conformal away from critical points we have that the vector field $g(z)/g'(z)$ is orthogonal to the level curves of g. Since $f'(z) = g(z) + zg'(z)$ and θ is a critical point of f, we see that the vector $-\theta$ is orthogonal to the level curve of g at θ. Note that $g'(\theta)$ is not zero because then $g(\theta) = 0$ and then $f(\theta) = 0$ which means that $\theta = 0$ since θ is a fixed point for f, and we have assumed that $\theta \neq p = 0$. Therefore the rays from 0 are transversal to the level curves of g near θ, and $|g(z)|$ increases in the direction of $-\theta$. Let us orient the level curves of g so that at θ the orientation agrees with the direction $-(\sqrt{-1})\theta$. Let k denote the curvature of the level curves of g with the given orientation. A calculation shows that

$$k = |g'(z)/g(z)|\,\mathbf{R}(1 - (g(z)g''(z)/(g'(z)^2)), \text{ see [2,p.359]}.$$

If we express this formula for k in terms of f and use that θ is both a fixed point and a critical point for f we obtain the formula

$$k = |1/\theta|\,\mathbf{R}(-1 - \theta f''(\theta)).$$

By hypothesis, $\mathbf{R}(\theta f''(\theta)) < 0$. Therefore, $k > -1/|\theta|$, which is the curvature of the circle C_θ with center 0 and which passes through θ. Since $-\theta$ is orthogonal to the g level curve at θ and $-\theta$ is also orthogonal to C_θ at θ, it follows that the g level curve is tangent to C_θ at θ. We conclude that in a small enough neighborhood of θ the level curve of g passing through θ is outside C_θ.

Suppose h is a perturbation of f. Define $j(z) = h(z)/z$. For sufficiently small perturbations h of f the level curves of j are transversal to the rays from 0, the level curve of j through σ is outside the circle C_σ which passes through σ with center 0. Let b denote the intersection of the ray through q and the level curve of j through σ. Let c denote the intersection of the ray through q and the circle C_σ. Note that $|j(z)|$ increases along the rays in the direction towards 0. Therefore $|j(c)| \geq |j(b)|$. By hypothesis, $|\sigma - 0| \geq |q - 0|$, so $|j(c)| \leq |j(q)|$. Since b and σ are on the same level curve of j, we conclude that $|j(\sigma)| \leq |j(q)|$. Since q is a fixed point for h we obtain $1 > |h(\sigma)/\sigma|$ or $|\sigma - 0| > |h(\sigma) - 0|$. Since we have assumed $p = 0$, this completes the proof of Theorem 1.

Remark. Theorem 1 is also true if all three inequality signs are reversed. The proof is analogous to the one given above.

Let us apply Theorem 1, and more particularly, the method of proof to the case of $f(z) = z^d + (d/(d-1))z$ and $p = 0$, which was the case discussed in the introduction. The critical points θ of f satisfy $\theta^{d-1} = -1/(d-1)$. For this polynomial $f(z)$ we find that $g(z) = z^{d-1} + d/(d-1)$ and $f''(\theta) = d(d-1)\theta^{d-2}$ which implies $\mathbf{R}(\theta f''(\theta)) < 0$. Without loss of generality we can restrict our attention to perturbations h of f which are degree d polynomials which satisfy $h(0) = 0$ and $h'(0) = d/(d-1)$. As was shown in [5], for any such perturbation h there is a critical point σ and a fixed point q near one of the critical points θ of f so that $|\sigma - 0| \geq |q - 0|$. From Thm. 1, we conclude that $|\sigma| > |h(\sigma)|$ and $S(h, 0, \sigma) < (d-1)/d$. This was already shown in [5]. Here we want to show in addition that

Theorem 2. *The stronger mean value conjecture (2) is true for any sufficiently small perturbation h of the above f with $z = 0$ and some critical point σ of h.*

Proof. Observe that the image by g of the level curve through θ is a circle centered at the origin. Since θ is fixed by f, $g(\theta) = 1$. Using the same notation as in the proof of Thm. 1, $g(C_{0\theta})$ is the circle of radius $1/(d-1)$ centered at $d/(d-1)$. For a perturbation h close to f, j will be a polynomial approximately the same as g. Therefore, $j(C_\sigma)$ is approximately a circle of radius $1/(d-1)$. So the curvature of $j(C_\sigma)$ is approximately $1/(d-1)$. Furthermore, $j(C_\sigma)$ is