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CODIMENSION ONE FOLIATIONS IN CP^n, $n \geq 3$, WITH KUPKA COMPONENTS

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1. Introduction

1.1 – Basic notions:

A codimension one holomorphic foliation in a complex manifold M can be given by an open covering $(U_\alpha)_{\alpha\in A}$ of M and two collections $(w\alpha)_{\alpha\in A}$ and $(g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi}$, such that:

(a) For each $\alpha \in A$, w_α is an integrable ($w_\alpha \wedge dw_\alpha = 0$) holomorphic 1-form in U_α, and $w_\alpha \not\equiv 0$.

(b) If $U_\alpha \cap U_\beta \neq \phi$ then $w_\alpha = g_{\alpha\beta} \cdot w_\beta$, where $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$.

Recall that $\mathcal{O}(V)$ is the set of holomorphic functions in V and $\mathcal{O}^*(V) = \{g \in \mathcal{O}(V) | g(p) \neq 0 \ \forall p \in V\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha\in A}, (w_\alpha)_{\alpha\in A,(g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})}$ be a foliation in M. The singular set of $\mathcal{F}$, $S(\mathcal{F})$, is by definition $S(\mathcal{F}) = \bigcup_{\alpha\in A} S_\alpha$, where $S_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0\}$. It follows from (a) and (b) that $S(\mathcal{F})$ is a proper analytic subset of M. The integrability condition implies that for each $\alpha \in A$ we can define a foliation $\mathcal{F}_\alpha$ (in the usual sense) in $U_\alpha - S_\alpha$, whose leaves are solutions of $w_\alpha = 0$. Condition (b) implies that if $U_\alpha \cap U_\beta \neq \phi$, then $\mathcal{F}_\alpha$ coincides with $\mathcal{F}_\beta$ in $U_\alpha \cap U_\beta - S(\mathcal{F})$. Hence we have a codimension one foliation defined in $M - S(\mathcal{F})$. A leaf of $\mathcal{F}$ is by definition, a leaf of this foliation.

If $S(\mathcal{F})$ has codimension one components, then it is possible to find a new foliation $\mathcal{F}_1 = ((U_\alpha)_{\alpha\in A}, (\tilde{w}_\alpha)_{\alpha\in A}, (\tilde{g}_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})$ such that $S(\mathcal{F}_1)$ has no components of codimension one, $S(\mathcal{F}_1) \subset S(\mathcal{F})$, and the leaves of $\mathcal{F}$ and $\mathcal{F}_1|(M - S(\mathcal{F}))$ are the same (in fact $w_\alpha = f_\alpha \cdot \tilde{w}_\alpha$, $f_\alpha \in \mathcal{O}(U_\alpha)$). From now on all the foliatons that we will consider *will not have codimension 1 singular components.*

1.2 – The Kupka set:

In 1964 I.Kupka proved the following result (see [**K**]);

1.2.1 THEOREM. *Let w be an integrable holomorphic 1-form defined in a neighborhood of $p \in \mathbf{C}^n$, $n \geq 3$. Supose that $w_p = 0$ and $dw_p \neq 0$. Then there exists a holomorphic coordinate system $(x, y, z_3, \ldots, z_n)$ defined in a neighborhood U of p such that $x(p) = y(p) = 0$ and $w = A(x,y)dx + B(x,y)dy$ in this coordinate system, where $A(0,0) = B(0,0) = 0$ and $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$.*

In fact Kupka proved this result in the real context, but his proof adapts very well in the holomorphic case.

1.2.2 Remarks: Let w, A, B and U be as in Theorem 1.2.1.

(i) The set $\{(x, y, z_3, \ldots, z_n) \in U | x = y = 0\} = V$ is containned in U. If the singular set S of w has no codimension 1 components, then V is a smooth codimension 2 piece of S and $(0,0)$ is an isolated solution of $A(x,y) = B(x,y) = 0$. By taking a smaller U if necessary we can suppose that $S \cap U = V$.

(ii) The foliation induced by $w = 0$ in U is equivalent to the product of the singular foliation in $U \cap \{z_3 = c_3, \ldots, z_n = c_n\} \subset \mathbf{C}^2 \times (c_3, \ldots, c_n)$ given by $Adx + Bdy = 0$ (or by the differential equation $\dot{x} = -B, \dot{y} = A$), by the codimension 2 foliation in U given by $x = c_1, y = c_2$. The singular set in this case is $V = \{x = y = 0\}$.

Let $\mathcal{F} = ((U_\alpha)_{\alpha\in A}, (w_\alpha)_{\alpha\in A}, (g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})$ be a foliation on M. We define the Kupka set of $\mathcal{F}$ by $K(\mathcal{F}) = \bigcup\limits_{\alpha\in A} K_\alpha$, where

$$K_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) \neq 0\}$$

Since $w_\alpha = g_{\alpha\beta} w_\beta$ in $U_\alpha \cap U_\beta \neq \phi$, we have $dw_\alpha = dg_{\alpha\beta} \wedge w_\beta + g_{\alpha\beta} dw_\beta$ which implies that $K_\alpha \cap U_\beta = K_\beta \cap U_\alpha$. It follows from (i) that $K(\mathcal{F})$ is a smooth complex codimension 2 submanifold of M. In fact $K(\mathcal{F}) = S(\mathcal{F}) - W(\mathcal{F})$ where $W(\mathcal{F}) = \bigcup_{\alpha \in A} W_\alpha$, $W_\alpha = \{p \in U_\alpha | w_\alpha(p) = 0 \text{ and } dw_\alpha(p) = 0\}$. Observe that $W(\mathcal{F})$ is an analytic subset of M.

1.2.3 Definition: We say that K is a *Kupka component* of $\mathcal{F}$ if K is an irreducible component of $S(\mathcal{F})$ and $K \subset K(\mathcal{F})$. Observe that a Kupka component of $\mathcal{F}$ is in particular a smooth connected codimension 2 analytic subset of M.

Let V be a connected codimension 2 submanifold of $K(\mathcal{F})$. It follows from the local product structure (see 1.2.1 and 1.2.2) that there exists a covering $(B_i)_{i \in I}$ of V by open sets of M, a collection of submersions $(\psi_i)_{i \in I}$, $\psi_i \colon B_i \to \mathbf{C}^2$, and a 1-form $w = A(x,y)dx + B(x,y)dy$ defined in a neighborhood C of $(0,0) \in \mathbf{C}^2$, such that:

(a) $\psi_i(B_i) \subset C$ for evere $i \in I$.

(b) $(0,0)$ is the unique singularity of w in C and $V \cap B_i = \psi_i^{-1}(0,0)$, for every $i \in I$.

(c) $\mathcal{F}|B_i$ is represented by $w_i^* = \psi_i^*(w)$.

We will say that $\mathcal{F}$ has *transversal type w or X* along V, where X is the vector field $-B\partial/\partial x + A\partial/\partial y$. The *linear transversal type of $\mathcal{F}$ along V* is, by definition, the linear part of X at $(0,0)$ in Jordan's canonical form, modulo multiplication by non-zero constants. Let L be the linear part of X at $(0,0)$ in Jordan's canonical form. We have the following possibilities:

(i) L is diagonal with eigenvalues $\lambda_1 \neq \lambda_2$.

(ii) L is diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.

(iii) L is not diagonal with eigenvalues $\lambda_1 = \lambda_2 \neq 0$.

Observe that, since $\frac{\partial B}{\partial x}(0,0) - \frac{\partial A}{\partial y}(0,0) \neq 0$, we have $tr(L) \neq 0$ and so the possibilities $\lambda_1 = \lambda_2 = 0$ or $\lambda_1 = -\lambda_2$ cannot occur.

In case (i) the two eigendirections of L induce via the submersions ψ_i, two line subbundles of the normal bundle $\nu(V)$ of V in M. We will call these line bundles L_1 (relative to λ_1) and L_2 (relative to λ_2). It is clear that $\nu(V) = L_1 \oplus L_2$. In case (iii) L has just one eigendirection which induces in the same way a line subbundle L_1 of $\nu(V)$. In the case of Kupka components we have the following (see [**G.M- L.N**]):

1.2.4 - THEOREM. *Let* $\dim(M) \geq 3$ *and* K *be a Kupka compact component of* $\mathcal{F}$. *We have:*

(a) In case (i), if $C(L_i)$ *is the first Chern class of* L_i, $i = 1, 2$, *considered in* $H^2(K, \mathbf{C})$, *then* $\lambda_1 C(L_2) = \lambda_2 C(L_1)$.
(b) In case (iii) we have $C(L_1) = 0$.
(c) In case (i), if $\lambda_2/\lambda_1 = p/q$, *where* $p, q \in \mathbf{Z}_+$ *are relatively primes and* $C(L_1) \neq 0$, *then* X *is linearizable.*

1.3 - Codimension 1 foliations of $\mathbf{C}P^n$, $n \geq 3$:

A holomorphic foliation in $\mathbf{C}P^n$ can be given by an integrable 1-form $w = \sum_{i=0}^{n} w_i dz_i$ ($w \wedge dw = 0$), with the following properties:

(a) $w_0, \ldots, w_n$ are homogeneous polynomials of the same degree ≥ 1.
(b) $i_R(w) = \sum_{i=0}^{n} w_i z_i \equiv 0$ ($R = \sum_{i=0}^{n} z_i \partial/\partial z_i$ is the radial vector field).

This form can be obtained as follows: let $\pi\colon \mathbf{C}^{n+1} - \{0\} \to \mathbf{C}P^n$ be the canonical projection and $\mathcal{F} = ((U_\alpha)_{\alpha\in A}, (w_\alpha)_{\alpha\in A}, (g_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})$ be a foliation in $\mathbf{C}P^n$. Let $\mathcal{F}^* = ((U^*_\alpha)_{\alpha\in A}, (w^*_\alpha)_{\alpha\in A}, (g^*_{\alpha\beta})_{U_\alpha\cap U_\beta\neq\phi})$ be the foliation in $\mathbf{C}^{n+1} - \{0\}$ defined by $U^*_\alpha = \pi^{-1}(U_\alpha), w^*_\alpha = \pi^*(w_\alpha)$ and $g^*_{\alpha\beta} = g_{\alpha\beta} \circ \pi$. Since for $U^*_\alpha \cap U^*_\beta \cap U^*_\gamma \neq \phi$ we have $g^*_{\alpha\beta} \cdot g^*_{\beta\gamma} \cdot g^*_{\gamma_\alpha} = 1$, we can use Cartan's solution of the multiplicative Cousin's problem in $\mathbf{C}^{n+1} - \{0\}$ (see [**G-R**]) to obtain an integrable 1-form η in $\mathbf{C}^{n+1} - \{0\}$ such that for any $\alpha \in A$, we have $\eta | U^*_\alpha = h_\alpha \cdot w^*_\alpha$, where $h_\alpha \in \mathcal{O}^*(U^*_\alpha)$. From Hartog's Theorem (see [**G-R**]), η extends to a holomorphic 1-form μ in $\mathbf{C}^{n+1}$. If $\mu = \mu_k + \mu_{k+1} + \ldots$ is the