CH. BONATTI
X. GÓMEZ-MONT

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# The Index of Holomorphic Vector Fields on Singular Varieties I[1]

Ch. Bonatti and X. Gómez-Mont

Given a complex analytic space $V$ with an isolated singuarity at $p$, there is a way to associate to a holomorphic vector field $X$ on $V$ an index at $p$ a la Poincaré-Hopf $\mathrm{Ind}(X, V, p)$ (see [Se],[GSV]). The objective of this series of papers is to understand this index. In the present paper we relate it to the $V$-multiplicity:

$$\mu_V(X,p) = \dim_{\mathbf{C}} \frac{\mathcal{O}_{\mathbf{C}^n,p}}{(f_1, \dots, f_\ell, X^1, \dots, X^n)}$$

where $f_1, \dots, f_\ell$ are generators of the ideal defining $V \subset \mathbf{C}^n$, $X^j$ are the coordinate functions of a holomorphic vector field that extends $X$ to a neighbourhood of 0 in $\mathbf{C}^n$ and the denominator denotes the ideal generated by the elements inside the parenthesis in the ring $\mathcal{O}_{\mathbf{C}^n,p}$ of germs of holomorphic functions at $p$. The main results are:

**Theorem 2.2.** *Let $(V, 0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n, 0)$ be an analytic space in the unit ball $\mathbf{B}_1$ which is smooth except for an isolated singularity at 0. Let $\Theta_r$ denote the Banach space of holomorphic vector fields on $V_r$ with continuous extensions to $\partial V_r$, $r < 1$, with its natural structure as an analytic space of infinite dimension. Then:*

*a) The function $V$-multiplicity at 0*

$$\mu_V(\ ,0)\colon \Theta_r \to \mathbf{Z}^+ \cup \{\infty\}$$

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*is upper semicontinuous and it is locally bounded at those points $X$ where $X$ has an isolated singularity on $V$ at 0.*

b) *The subsets of $\Theta_r$ defined by $\mu(\ ,0) \geq K$ are analytic subspaces and the minimum value of $\mu_V(\ ,0)$ in $\Theta_r$ is attained on an open dense subset $\tilde{\Gamma}_1$ of $\Theta_r$.*

c) *The subset of $\Theta_r$ formed by vector fields whose critical set at 0 has positive dimension is an analytic subspace of $\Theta_r$.*

We introduce the Euler characteristic $\chi_V(X,0)$ of $X \in \Theta_r$ at 0 in (2.10) and show:

**Theorem 2.5.** *For $X \in \Theta_r$ with an isolated singularity at 0, $s \ll r$ and $0 \ll \varepsilon$, we have:*

1) *For any family of vector fields $\{X_t\}_{t\in T}$, parametrized by a finite dimensional analytic space $(T,0) \to (\Theta_r, X)$ such that the $V$- multiplicity at 0 of the general vector field $X_t$ of the family is minimal $\mu_V$, we have:*

$$\chi_V(X,0) = \chi_0^{\text{tor}}(\mathcal{O}_{z_{T,s}}, \mathcal{O}_{\{X\}})$$

*where the right and hand side is the Euler characteristic of higher torsion groups.*

2) *For $Z \in U(X,\varepsilon)$ we have*

$$\chi_V(X,0) = \chi_V(Z,0) + \sum_{\substack{Z(p_j)=0 \\ p_j\in V_s-\{0\}}} \mu_V(Z,p_j)$$

3) *For $X \in \Theta_r$ with an isolated critical point at 0, we have:*

$$0 < \chi_V(X,0) \leq \mu_V(X,0)$$

*and $\chi_V(X,0) = \mu_V(X,0)$ if and only if the universal critical set $\mathcal{Z}_r$ is $\pi_1$-anaflat at $(X,0)$ (in particular this happens in $\tilde{\Gamma}_1$).*

Let $X \in \Theta_r$, we say that the *critical set of $X$ does not bifurcate* if there is $\varepsilon > 0$ and $s > 0$ such that for $Y \in \mathbf{U}(X,\varepsilon) \subset \Theta_r$ we have that the only critical point of $Y$ on $V_s$ is 0, (that is, $X$ has an isolated singularity at 0 as well as any sufficiently near vector field in $\Theta_r$ and there is no other critical point uniformly in a neighbourhood $V_s$ of 0).

**Theorem 2.6.** *Let $(V,0) \subset \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then the set of points in $\Theta_r$ whose critical set does not bifurcate contains the connected dense open subset $\tilde{\Gamma}_1 \subset \Theta_r$ consisting of vector fields with minimum $V$-multiplicity.*

**Theorem 3.1.** *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is an integer $K$ such that*

$$Ind_W(X,V,0) = \chi_V(X,0) + K$$

*for $X$ in the dense open set $\Theta'$ of vector fields in $\Theta_r$ with an isolated singularity at 0. For $X$ in the dense open set of $\Theta'$ where the universal critical set $\mathcal{Z}_r$ is $\Theta_r$-anaflat we have*

$$Ind_W(X,V,0) = \mu_V(X,0) + K$$

**Corollary 3.2.** *Let $(V,0) \subseteq \mathbf{B}_1 \subset (\mathbf{C}^n,0)$ be an analytic space which is smooth except for an isolated singularity at 0, then there is a constant $L$ such that $Ind_W(X,V,0) \geq L$ for every germ of holomorphic vector field $X$ on $V$ with an isolated singularity at 0 on $V$.*

In the first section we analyse the index on smooth compact manifolds with boundary. We prove:

**Proposition 1.1.** *Let $X$ and $Y$ be $C^1$-vector fields defined on the compact manifold with boundary $(W,\partial W)$ and non-vanishing on $\partial W$ and let $[\Gamma_X]$ denote the fundamental class of the graph of $X/\|X\|$ on the sphere bundle $\mathbf{S}$ of unit tangent vectors of $W$ restricted to $\partial W$ (with respect to some Riemannian metric on $W$). Then*

$$Ind(X,\partial W,W) - Ind(Y,\partial W,W) = [\Gamma_X]\cdot[\Gamma_{-Y}]$$

*where we do the intersection in homology of* **S**.

In the second section we develop the properties of the $V$-multiplicity, and in the third we compare the $V$-multiplicity with the topological index.

## 1. The index of vector fields on manifolds with boundary

Let $W$ be a compact oriented manifold of dimension $m$ with boundary, $\partial W$, oriented in the natural way. Given a never vanishing $C^0$-vector field $X$ in a neighbourhood of $\partial W$, the *index of* $X$ *on the boundary of* $W$, $\mathrm{Ind}(X, \partial W, W)$ may be defined by extending $X$ to a vector field $\tilde{X}$ on $W$ with isolated singularities, and then adding up the indices at the singularities of $\tilde{X}$. The index is independent of the chosen extension $\tilde{X}$ (see [Mi],[Se]).

To understand the dependence of the index on the manifold $W$, we will prove that the difference of the indices of 2 vector fields may be computed exclusively in terms of boundary data:

**Proposition 1.1.** *Let $X$ and $Y$ be $C^1$-vector fields defined on the compact manifold with boundary $(W, \partial W)$ and non-vanishing on $\partial W$ and let $[\Gamma_X]$ denote the fundamental class of the graph of $X/\|X\|$ on the sphere bundle* **S** *of unit tangent vectors of $W$ restricted to $\partial W$ (with respect to some Riemannian metric on $W$). Then*

$$Ind(X, \partial W, W) - Ind(Y, \partial W, W) = [\Gamma_X] \cdot [\Gamma_{-Y}]$$

*where we do the intersection in homology of* **S**.

**Proof.** Since the index and the fundamental classes do not change if we make a small perturbation, we will assume that $X$ and $Y$ are in general position. Namely we will assume that if the zeroes $\mathcal{Z} \subset \mathbf{C} \times W$ of the vector fields $\{X_t = (1-t)X + tY\}_{t\in[0,1]}$ intersect $\partial W$, say at $0_t$, then at $0_t$: $X_t$ has a zero