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HOLOMORPHIC GERBES AND THE BEILINSON REGULATOR

by Jean-Luc BRYLINSKI

Introduction

For X a smooth complex projective variety, Beilinson has defined regulator maps $c_{m,i} : K_i(X) \to H^{2m-i}(X, \mathbb{Z}(m)_D)$ from algebraic K-theory to Deligne cohomology [**Be1**]. For a variety over $\mathbb{Q}$, the conjectures of Beilinson express the leading term of the expansion of the Hasse-Weil L-functions at an integer in terms of this regulator.

Many computations of the regulator have been performed by Beilinson himself [**Be1**] [**Be2**] and by other authors [**D-W**] [**Ra2**]. There are however few cases where the regulator map has been described geometrically, the main reason being that the Deligne cohomology groups themselves do not have an easy global geometric interpretation. There is an important case where a geometric interpretation has been obtained by Bloch, Deligne and Ramakrishnan, namely that of the regulator $c_{2,2} : K_2(X) \to H^2(X, \mathbb{Z}(2)_D)$. For X projective, this goes as follows: Deligne showed that the group $H^2(X, \mathbb{Z}(2)_D)$ identifies with the group of isomorphism classes of holomorphic line bundles over X equipped with a holomorphic connection. Then Bloch [**Bl**] and Deligne [**De2**], constructed a holomorphic line bundle associated to a pair of invertible holomorphic functions, and Bloch showed that this gives a regulator map from $K_2(X)$ to the group of isomorphism classes of holomorphic line bundles with connection. Ramakrishnan gave an interpretation of this construction in terms of the three-dimensional Heisenberg group [**Ra1**].

Line bundles can however be used only in describing this special case of the Beilinson regulator. Other Deligne cohomology groups are in fact related to higher analogs of line bundles, which are called gerbes [**G**] (with band the sheaf of invertible holomorphic functions), 2-gerbes [**Bre**], etc.... In this paper, we give a geometric description for the regulator map $c_{2,1} : K_1(X) \to H^3(X, \mathbb{Z}(2)_D)$. This is based on the interpretation of the Deligne cohomology group $H^3(X, \mathbb{Z}(2)_D)$ as the group of equivalence classes of holomorphic gerbes equipped with a holomorphic connective structure. These notions were developed in [**Bry1**] and [**Bry2**] where they were applied to the geometry of loop spaces and of the space of knots in a three-manifold. For X projective, the Deligne cohomology group in question is the quotient of the dual of $H_2(X, \mathbb{R})$ by the linear forms of the type $\gamma \mapsto \Im(\int_\gamma \omega)$, for ω a holomorphic 2-form, and $\Im$ denotes the imaginary part. In terms of holomorphic gerbes, the linear form on $H_2(X, \mathbb{R})$ thus obtained is the *holonomy* of the holomorphic gerbe. In fact, the geometric significance of gerbes is that they give rise to such holonomy functionals for mappings of surfaces into the ambient manifold.

Underlying this is a theory of curvings compatible with the holomorphic structure of a gerbe. These curvings which are flat (i.e., have zero 3-curvature) are unique precisely up to a holomorphic 2-form; this is our geometric explanation for the ambiguity of a holomorphic 2-form in the regulator map.

In principle such ideas will lead to a geometric description of all regulator maps, once the categorical aspects have been cleared up. Hopefully this would lead to a better understanding of algebraic K-theory itself.

To make this paper self-contained, we have included a discussion of holomorphic gerbes and their differential geometry (connective structure, curving and 3-curvature). We only discuss Deligne cohomology, as opposed to the more delicate Deligne-Beilinson cohomology, except for some comments related to growth structures on gerbes, at the end of §2.

Finally in §5 we give the geometric description of the regulator $K_1(X) \to H^3(X, \mu_m^{\otimes 2})$ for m odd. This uses an analog of the line bundle

(f, g) of Bloch and Deligne, associated to a pair f, g of invertible regular functions. This analog is a gerbe, which is the obstruction to lifting an abelian covering with group $\mu_m \times \mu_m$ to a covering whose group is a finite Heisenberg group. The obstruction vanishes when $g = 1 - f$, due to the existence of an embedding of $\mu_m^{\otimes -1}$ into the jacobian of the Fermat curve $x^m + y^m = 1$. This is closely related to work of Deligne [**De3**] and Ihara [**I**].

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1. Construction of holomorphic gerbes

We recall the notion of *gerbe* $\mathcal{C}$ on a space X, with band a commutative sheaf of groups A. This is a sheaf of categories (or stack) over X in the following sense. For every continuous map $f : Y \to X$, which is a local homeomorphism, there is category $\mathcal{C}(Y \xrightarrow{f} X)$ (or simply $\mathcal{C}(Y)$). Given a diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$ of local homeomorphisms, there is a pull-back functor $g^* : \mathcal{C}(Y \xrightarrow{f} X) \to \mathcal{C}(Z \xrightarrow{gf} X)$. We do not require that $(hg)^* = g^* h^*$, since this does not hold in geometric situations. We do assume that there is a given invertible natural transformation $\theta_{g,h} : g^* h^* \xrightarrow{\sim} (hg)^*$, such that some commutative diagram commutes, for any diagram $V \xrightarrow{k} W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X$ (we refer the reader to [**Bry2**] for details). Given a diagram of local homeomorphisms $Z \xrightarrow{g} Y \xrightarrow{f} X$, with g surjective, one has a descent category, whose objects are pairs (P, ϕ), where P is an object of $\mathcal{C}(Z)$, and $\phi : p_1^* P \xrightarrow{\sim} p_2^* P$ is an isomorphism between objects of $\mathcal{C}(Z \times_Y Z)$. We say that $\mathcal{C}$ is a *sheaf of categories* if the natural functor from $\mathcal{C}(Y)$ to the above descent category is an equivalence of categories, for any such diagram $Z \xrightarrow{g} Y \xrightarrow{f} X$.

Let now A be a sheaf of abelian groups over X. A *gerbe* over X with band A is a sheaf of categories $\mathcal{C}$ over X, together with an isomorphism $\alpha_P : A \tilde{\to} \underline{Aut}(P)$ for any object P of $\mathcal{C}(Y)$, such that the following properties are satisfied:

(1) The categories $\mathcal{C}(Y)$ are groupoids (i.e., every morphism is invertible).

(2) The isomorphisms α_P commute with all morphisms in $\mathcal{C}$.

(3) Two objects of $\mathcal{C}(Y)$ are locally isomorphic.

(4) There exists a surjective local homeomorphism $f : Y \to X$ such that $\mathcal{C}(Y)$ is non-empty.

A gerbe on X with band A leads to a cohomology class in $H^2(X, A)$. In fact, Giraud proved [**G**] that $H^2(X, A)$ identifies with the group of equivalence classes of gerbes over X with band A.

We will study gerbes over a complex-analytic manifold X, with band equal to the sheaf of groups $\mathcal{O}_X^*$. Such gerbes will be called *holomorphic gerbes*.

Recall briefly how a divisor D on X leads to a line bundle $\mathcal{O}(D)$. Here $D = \sum_i n_i D_i$ is a formal combination, with integer coefficients, of irreducible subvarieties of codimension one. There are several descriptions of $\mathcal{O}(D)$. First we can describe the space of sections $\Gamma(U, \mathcal{O}(D))$ for any open set U, as comprised of all meromorphic functions f on U such that $div(f) + D \geq 0$ in U, where $div(f)$ is the divisor of f. If we wish merely to describe the class of $\mathcal{O}(D)$ in $Pic(X) = H^1(X, \mathcal{O}_X^*)$, we may use the exact sequence of sheaves on X

$$0 \to \mathcal{O}_X^* \to \mathcal{O}_X(*Y)^* \xrightarrow{v_Y} \nu_{Y\,*}\mathbb{Z}_{\tilde{Y}} \to 0 \qquad (1-1)$$

where $Y = \cup_{i:n_i \neq 0} D_i$ is the support of D, $\nu_Y : \tilde{Y} \to Y$ is the normalization of Y. Note that $\tilde{Y} = \coprod_i \tilde{D}_i$, where $\tilde{D}_i$ is the normalization of D_i. The sheaf of algebras $\mathcal{O}_X(*Y)$ is the direct limit of the $\mathcal{O}_X(n \cdot Y)$ for $n \geq 1$; in other words, a section of $\mathcal{O}_X(*Y)$ over an open subset U of X is a holomorphic function on $U \setminus Y$, which is meromomorphic over U. The homomorphism v_Y associates to a meromorphic function f its polar divisor. More precisely,